SPECIAL BIRATIONAL TRANSFORMATIONS OF TYPE $(2,1)$

BAOHUA FU AND JUN-MUK HWANG

ABSTRACT. A birational transformation $\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$, where $Z \subset \mathbb{P}^N$ is a nonsingular variety of Picard number 1, is called a *special birational transformation of type* (a,b) if Φ is given by a linear system of degree a, its inverse Φ^{-1} is given by a linear system of degree b and the base locus $S \subset \mathbb{P}^n$ of Φ is irreducible and nonsingular. In this paper, we classify special birational transformations of type $(2,1)$. In addition to previous works [AS] and [R2] on this topic, our proof employs natural $\mathbb{C}^*$-actions on Z in a crucial way. These $\mathbb{C}^*$-actions also relate our result to the problem studied in [FH].

1. Introduction

Recall (e.g. Section 2 in [AS] or Definition 4.1 in [R2]) that a birational transformation

$$\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$$

where $Z \subset \mathbb{P}^N$ is a nonsingular projective variety of Picard number 1 is called a *special birational transformation of type* (a,b) if

(1) the base locus $S \subset \mathbb{P}^n$ of Φ is irreducible and nonsingular;
(2) the rational map Φ is given by a linear system belonging to $\mathcal{O}_{\mathbb{P}^n}(a)$; and
(3) the inverse rational map Φ^{-1} is given by a linear system belonging to $\mathcal{O}_Z(b)$.

When $Z = \mathbb{P}^n$, this is a special Cremona transformation, a classical topic in projective algebraic geometry. It is a challenging problem to classify special birational transformations. Even for special Cremona transformations, a complete classification is still missing. Special Cremona transformations of type $(2,2)$ have been classified by Ein and Shepherd-Barron in [ES] by relating them to Severi varieties classified by Zak ([Z]). In [R2], special Cremona transformations of types $(2,3)$ and $(2,5)$ have been classified. Recently Alzati and Sierra ([AS]) have

Baohua Fu is supported by National Scientific Foundation of China (11225106 and 11321101).

extended [ES] to a classification of special birational transformations of type $(2,2)$ for a wider class of Z.

In this paper, we will give a complete classification of special birational transformations of type $(2,1)$. This classification can be described in terms of the classification of the base locus $S \subset \mathbb{P}^n$, which is contained in a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ for the type $(2,1)$. Our main result is the following.

Theorem 1.1. *The base locus $S^d \subset \mathbb{P}^{n-1}$ of a special birational transformations of type $(2,1)$ is projectively equivalent to one of the following:*

(a) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ *for* $d \geq 1$*;*
(b) $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ *for* $d \geq 3$*;*
(c) *the 6-dimensional Grassmannian* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$*;*
(d) *the 10-dimensional Spinor variety* $\mathbb{S}_5 \subset \mathbb{P}^{15}$*;*
(e) *a nonsingular codimension* ≤ 2 *linear section of* $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$*;*
(f) *a nonsingular codimension* ≤ 3 *linear section of* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$.

The description of the varieties in Theorem 1.1 as well as a more precise formulation of the result will be given in Section 2. Our proof of Theorem 1.1, to be given in Section 6, uses previous works on this topic in [AS] and [R2]. The main strategy is an inductive argument on VMRT (see Proposition 4.6) developed by Russo in [R2]. There are two new ingredients in our approach: the use of natural $\mathbb{C}^*$-actions on Z, which reveals topological relations between the base loci of Φ and Φ^{-1}, and a study of the intersection of entry loci on the base locus of Φ, which exhibits a delicate structure in the projective geometry of the base locus. The part on $\mathbb{C}^*$-action is presented in Section 3 and the part on the intersection of entry loci is presented in Section 5. The use of $\mathbb{C}^*$-actions on Z is motivated by our previous work [FH] on projective manifolds with nonzero prolongations. As a matter of fact, we will see in Section 7 that a prime Fano manifold Z is the target of a special birational transformation of type $(2,1)$ if and only if it has nonzero prolongation. By this correspondence, we can use Theorem 1.1 to give a new proof (see Theorem 7.13) of the main classification result of [FH]. This new proof corrects an error in the classification in [FH], as explained in Remark 7.14.

We will work over complex numbers. For simplicity, a nonsingular irreducible variety will be called a manifold.

Acknowledgments: We are grateful to José Carlos Sierra for explaining to us the examples in Proposition 2.11 and Proposition 2.12 and for

a careful reading of the draft. We would like to thank Francesco Russo for pointing out a mistake in a previous version and for numerous suggestions. We are grateful to Qifeng Li for discussions on Section 5 and to Fyodor Zak for suggesting the argument in Remark 2.13 (3).

2. Statement of the classification of special quadratic manifolds

Definition 2.1. Let U be a vector space and let $\sigma : \mathrm{Sym}^2 U \to W$ be a surjective linear map to a vector space W. Let us denote by $W^* \subset \mathrm{Sym}^2 U^*$ the annihilator of $\mathrm{Ker}(\sigma)$, which is naturally dual to W.

(1) Let $\psi^o : \mathbb{P}U \dashrightarrow \mathbb{P}W$ be the rational map defined by the linear system $W^* \subset \mathrm{Sym}^2 U^* \simeq H^0(\mathbb{P}U, \mathcal{O}(2))$. The scheme-theoretic base locus of ψ^o will be denoted by $B(\sigma) \subset \mathbb{P}U$ and the proper image of $\mathbb{P}U$ under ψ^o will be denoted by $Y(\sigma) \subset \mathbb{P}W$.

(2) Fix a 1-dimensional vector space T with a fixed identification $T = \mathbb{C}$. Define a rational map $\phi^o : \mathbb{P}(T \oplus U) \dashrightarrow \mathbb{P}(T \oplus U \oplus W)$ by

$$[t : u] \mapsto [t^2 : tu : \sigma(u,u)] \text{ for } t \in T, u \in U.$$

The proper image of $\mathbb{P}(T \oplus U)$ under ϕ^o will be denoted by $Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$. The scheme-theoretic base locus of ϕ^o coincides with $B(\sigma) \subset \mathbb{P}U = \mathbb{P}(0 \oplus U)$.

We will say that σ is a *special system of quadrics* if (a) the base locus subscheme $B(\sigma) \subset \mathbb{P}U$ is irreducible and nonsingular and (b) the image $Z(\sigma)$ of ϕ^o is nonsingular. A projective manifold $S \subset \mathbb{P}U$ is called a *special quadratic manifold* if $S = B(\sigma)$ for a special system of quadrics $\sigma : \mathrm{Sym}^2 U \to W$.

In this case, the rational map ψ^o comes from a morphism

$$\psi : \mathrm{Bl}_S(\mathbb{P}U) \to Y(\sigma) \subset \mathbb{P}W$$

where $\mathrm{Bl}_S(\mathbb{P}U)$ is the blow-up of $\mathbb{P}U$ along S. We will denote by $F \subset \mathrm{Bl}_S(\mathbb{P}U)$ its exceptional divisor. The rational map ϕ^o comes from a morphism

$$\phi : \mathrm{Bl}_S(\mathbb{P}(T \oplus U)) \to Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$$

where $\mathrm{Bl}_S(\mathbb{P}(T \oplus U))$ is the blow-up of $\mathbb{P}(T \oplus U)$ along $S \subset \mathbb{P}U = \mathbb{P}(0 \oplus U)$. We will denote by $E \subset \mathrm{Bl}_S(\mathbb{P}(T \oplus U))$ its exceptional divisor.

We have the commuting diagram

$$\begin{array}{ccccccc} F & \subset & \mathrm{Bl}_S(\mathbb{P}U) & \xrightarrow{\psi} & Y(\sigma) & \subset & \mathbb{P}W \\ \cap & & \cap & & \cap & & \cap \\ E & \subset & \mathrm{Bl}_S(\mathbb{P}(T \oplus U)) & \xrightarrow{\phi} & Z(\sigma) & \subset & \mathbb{P}(T \oplus U \oplus W). \end{array}$$

Note that ϕ sends $U \cong \mathbb{P}(T \oplus U) \backslash \mathbb{P}U$ isomorphically to $Z(\sigma) \backslash \mathbb{P}(U \oplus W)$. Thus $Z(\sigma)$ is a rational variety and $\phi^o : \mathbb{P}(T \oplus U) \dashrightarrow Z(\sigma)$ is a birational map.

It follows immediately from the definition that for a special quadratic manifold $S \subset \mathbb{P}U$, the birational map $\phi^o : \mathbb{P}(T \oplus U) \dashrightarrow Z(\sigma)$ is a special birational transformation of type $(2, 1)$ in the sense of Section 1. Conversely we have

Proposition 2.2. *Let $\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$ be a special birational transformation of type $(2, 1)$ as defined in Section 1. Then the base locus S of Φ is contained in a hyperplane $\mathbb{P}^{n-1}$ of $\mathbb{P}^n$, the subvariety $S \subset \mathbb{P}^{n-1}$ is a special quadratic manifold and Φ coincides with the birational map ϕ^o for a special system of quadrics $\sigma : \mathrm{Sym}^2 U \to W$.*

Proof. By Proposition 2.3 (a) [ES], the base locus $S \subset \mathbb{P}^n$ of Φ is contained in a hyperplane. Since S is defined by quadratic equations, $S \subset \mathbb{P}^{n-1}$ is also defined by quadratic equations. As S is the base locus of Φ, the rational map Φ is given by a linear subspace $W^* \subset H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$ and the full linear system $H^0(\mathbb{P}^n, \mathcal{I}_{\mathbb{P}^{n-1}}(2))$. Then the map ϕ^o associated to W coincides with Φ. □

Example 2.3. We list some homogeneous examples of special quadratic manifolds in the next table. The data in this table can be found from Theorem 3.8 in Chapter III of [Z]. Note that in all these examples, the dimension a of W is equal to $h^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$, namely, we have $W^* = H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$.

S	$\mathbb{Q}^d$	$\mathbb{P}^1 \times \mathbb{P}^{d-1}$	$\mathrm{Gr}(2, 5)$	$\mathbb{S}_5$
$S \subset \mathbb{P}^{n-1}$	$\mathcal{O}(1)$	Segre	Plücker	Spinor
Y	point	$\mathrm{Gr}(2, d)$	$\mathbb{P}^4$	$\mathbb{Q}^8$
$Y \subset \mathbb{P}^{a-1}$	identity	Plücker	identity	$\mathcal{O}(1)$
Z	$\mathbb{Q}^{d+2}$	$\mathrm{Gr}(2, d+2)$	$\mathbb{S}_5$	$\mathbb{OP}^2$
$Z \subset \mathbb{P}^{n+a}$	$\mathcal{O}(1)$	Plücker	Spinor	Severi
$d = \dim S$	$d = n - 2$	d	6	10
$n = \dim Z$	$n = d + 2$	$2d$	10	16
$m = \dim Y$	0	$2(d-2)$	4	8
a	1	$d(d-1)$	5	10

Example 2.4. Here we give some examples of special birational transformations of type $(2,1)$ with $W^* \subsetneq H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$. Recall from Example 2.3 that the special quadratic manifold $S = \mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ with $d \geq 6$ is associated to the special system of quadrics

$$\sigma : \mathrm{Sym}^2 U \to W = H^0(\mathbb{P}U, \mathcal{I}_S(2))^*.$$

The corresponding birational transformation is

$$\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z := \mathrm{Gr}(2, d+2) \subset \mathbb{P}(T \oplus U \oplus W)$$

with $Y = \mathrm{Gr}(2,d) \subset \mathbb{P}W = \mathbb{P}(\wedge^2 \mathbb{C}^d)$. Note that $\mathrm{Sec}(Z) \cap \mathbb{P}W = \mathrm{Sec}(Y)$. Take any linear subspace $L \subset W$ such that $\mathbb{P}L \cap \mathrm{Sec}(Y) = \emptyset$, then $\mathbb{P}L \cap \mathrm{Sec}(Z) = \emptyset$. Let

$$p_L : \mathbb{P}(T \oplus U \oplus W) \dashrightarrow \mathbb{P}(T \oplus U \oplus W/L)$$

be the projection from $\mathbb{P}L$. Then p_L sends Z (resp. Y) isomorphically to a subvariety

$$Z_L \subset \mathbb{P}(T \oplus U \oplus W/L) \text{ (resp. } Y_L \subset \mathbb{P}(W/L)).$$

The map

$$\phi_L^\circ := p_L \circ \phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z_L$$

is a special birational transformation of type $(2,1)$ associated to the special system of quadrics

$$\sigma_L : \mathrm{Sym}^2 U \to W/L.$$

with $Y(\sigma_L) = Y_L \subset \mathbb{P}(W/L)$ and $B(\sigma_L) \subset \mathbb{P}U$ being $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$.

To discuss non-homogeneous examples, it is convenient to introduce the following notion.

Definition 2.5. Let $Z \subset \mathbb{P}V$ be a nondegenerate submanifold and let $W \subset V$ be a subspace such that $\mathbb{P}W \subset Z$. Denote by $(V/W)^* \subset V^*$ the set of linear functionals on V annihilating W such that $\mathbb{P}(V/W)^*$ parameterizes the set of hyperplanes in $\mathbb{P}V$ containing $\mathbb{P}W$. Then a general member of $\mathbb{P}(V/W)^*$ is called a $\mathbb{P}W$-*general* hyperplane in $\mathbb{P}V$. More generally, a linear subspace of codimension-s in $\mathbb{P}V$ is $\mathbb{P}W$-*general* if it is defined by a general member of $\mathrm{Gr}(s, (V/W)^*)$, i.e., it is general among subspaces of codimension-s containing $\mathbb{P}W$.

Proposition 2.6. *Let $S = B(\sigma) \subset \mathbb{P}U$ be a special quadratic manifold defined by $\sigma : \mathrm{Sym}^2 U \to W$. Assume that $\dim S \geq 2$, $\dim \mathbb{P}U > \dim Y(\sigma)$ and the intersection of $Z(\sigma)$ with a $\mathbb{P}W$-general hyperplane of $\mathbb{P}(T \oplus U \oplus W)$ is nonsingular. Then for a general subspace $U' \subset U$ of codimension 1, the restriction $\sigma' : \mathrm{Sym}^2 U' \to W$ of σ is a special system of quadrics such that*

(i) *the base locus scheme* $B(\sigma') \subset \mathbb{P}U'$ *coincides with the hyperplane section* $S' = S \cap \mathbb{P}U' \subset \mathbb{P}U$ *of* $S = B(\sigma)$*;*
(ii) $Y(\sigma') = Y(\sigma) \subset \mathbb{P}W$*; and*
(iii) $Z(\sigma') = Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$.

Proof. Firstly, we claim that for a general $U' \subset U$, the hyperplane section $Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$ is nonsingular. This is a consequence of the assumption that a $\mathbb{P}W$-general hyperplane section of $Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$ is nonsingular. To see this, associate to each vector $v \in U$ the linear automorphism g_v of $\mathbb{P}(T \oplus U \oplus W)$ defined by

$$g_v : [t : u : w] \mapsto [t : u + tv : w + 2\sigma(u, v) + t\sigma(v, v)].$$

For a general choice of $v \in U$ and $U' \subset U$, the automorphism g_v sends $\mathbb{P}(T \oplus U' \oplus W)$ to a $\mathbb{P}W$-general hyperplane of $\mathbb{P}(T \oplus U \oplus W)$. The g_v-image of a general point $[1 : u : \sigma(u, u)] \in Z(\sigma)$ is

$$\begin{aligned} g_v([1 : u : \sigma(u, u)]) &= [1 : u + v : \sigma(u, u) + 2\sigma(u, v) + \sigma(v, v)] \\ &= [1 : u + v : \sigma(u + v, u + v)] \in Z(\sigma). \end{aligned}$$

Thus g_v preserves $Z(\sigma)$. Consequently, for a general choice of v, the automorphism g_v sends the hyperplane section $Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$ to a $\mathbb{P}W$-general hyperplane section of $Z(\sigma)$. This proves the claim.

Now we can choose a general subspace $U' \subset U$ of codimension 1 such that

(1) $\dim \mathbb{P}U' \geq \dim Y(\sigma)$ and $\dim U - \dim U' < \dim S$;
(2) the (scheme-theoretic) linear section $S' := S \cap \mathbb{P}U'$ is nonsingular; and
(3) the restriction $\psi^o|_{\mathbb{P}U'}$ is dominant over $Y(\sigma)$.

Then $\sigma' : \mathrm{Sym}^2 U' \to W$ is surjective, and $S' = B(\sigma') \subset \mathbb{P}U'$ is a special quadratic manifold with $Y(\sigma') = Y(\sigma)$, and $Z(\sigma') = Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$. ☐

To check the condition in Proposition 2.6, we need the following.

Proposition 2.7. *In the setting of Definition 2.5, let* $Z \subset \mathbb{P}V$ *be a nondegenerate submanifold containing* $\mathbb{P}W$*. Define* $Z^*_W \subset \mathbb{P}V^*$ *by*

$$Z^*_W = \{[H] \in \mathbb{P}(V/W)^* \mid H \cap Z \textit{ is singular at a point of } \mathbb{P}W\}.$$

Assume that $\dim Z^*_W < \dim \mathbb{P}(V/W)^*$*. Then for a* $\mathbb{P}W$*-general hyperplane* $[H] \in \mathbb{P}(V/W)^*$*, the intersection* $\mathcal{Z} := Z \cap H$ *is a nonsingular subvariety containing* $\mathbb{P}W$ *and the submanifold* $\mathcal{Z} \subset H$ *satisfies* $\dim \mathcal{Z}^*_W \leq \dim Z^*_W$.

Proof. The intersection of Z with a $\mathbb{P}W$-general H is nonsingular outside $\mathbb{P}W$ by Bertini. But we can choose $[H] \in \mathbb{P}(V/W)^*$ outside Z^*_W

by the dimension condition $\dim Z^*_W < \dim \mathbb{P}(V/W)^*$. Thus $\mathcal{Z} = Z \cap H$ is nonsingular and it contains $\mathbb{P}W$. It remains to check $\dim \mathcal{Z}^*_W \leq \dim Z^*_W$.

By definition,

$$\mathcal{Z}^*_W = \{[L] \in \mathbb{P}(H/W)^* \mid L \cap \mathcal{Z} \text{ is singular at a point of } \mathbb{P}W\}.$$

Given an element $[L]$ of $\mathcal{Z}^*_W$, then $L \cap \mathcal{Z}$ is singular at some point say $x \in \mathbb{P}W$. Take the hyperplane $\widetilde{L} \subset V$ to be the linear span of L and $T_x Z$, then $L = \widetilde{L} \cap H$ and $\widetilde{L} \cap Z$ is singular at $x \in \mathbb{P}W$. This shows that the image of Z^*_W under the projection $\mathbb{P}(V/W)^* \setminus \{H\} \to \mathbb{P}(\hat{H}/W)^*$ contains $\mathcal{Z}^*_W$, where $\hat{H}$ is the hyperplane in V corresponding to H. This implies that $\dim \mathcal{Z}^*_W \leq \dim Z^*_W$. □

By applying Proposition 2.7 repeatedly, we have the following.

Corollary 2.8. *In Proposition 2.7, let s be a positive integer satisfying*

$$s < \dim Z \text{ and } \dim Z^*_W \leq \dim \mathbb{P}(V/W)^* - s.$$

Then a $\mathbb{P}W$-general linear section of Z with codimension s is nonsingular.

Let us recall the following results from Proposition 2.19 and Remark 2.20 in Chapter III of [Z].

Proposition 2.9. *(i) Let $Y = \mathbb{P}W \subset Z \subset \mathbb{P}V$ be $\mathbb{P}^2 \subset \mathrm{Gr}(2,5) \subset \mathbb{P}^9$ from $S = \mathbb{P}^1 \times \mathbb{P}^2$ of Example 2.3. Then Z^*_W is isomorphic to a cone over S. In particular,* $\dim \mathbb{P}(V/W)^* - \dim Z^*_W = 6 - 4 = 2$.

(ii) Let $Y = \mathbb{P}W \subset Z \subset \mathbb{P}V$ be $\mathbb{P}^4 \subset \mathbb{S}_5 \subset \mathbb{P}^{15}$ from $S = \mathrm{Gr}(2,5)$ of Example 2.3. Then Z^*_W is isomorphic to a cone over S. In particular, $\dim \mathbb{P}(V/W)^* - \dim Z^*_W = 10 - 7 = 3$.

Lemma 2.10. *Recall that a subvariety $S \subset \mathbb{P}^{n-1}$ is* arithmetically Cohen-Macaulay *if $H^i(\mathbb{P}^{n-1}, \mathcal{I}_S(l)) = 0$ for all l and for all $0 < i <$* $\dim(S) + 1$. *Let $S \subset \mathbb{P}^{n-1}$ be either the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or the Plücker embedding* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. *Let $S' \subset \mathbb{P}^{n-s-1}$ be any nonsingular linear section of $S \subset \mathbb{P}^{n-1}$ with codimension $s < \dim S$. Then $S' \subset \mathbb{P}^{n-s-1}$ is arithmetically Cohen-Macaulay and $H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2)) = H^0(\mathbb{P}^{n-s-1}, \mathcal{I}_{S'}(2))$.*

Proof. By [Z] (Chapter III, Theorem 1.2), $S \subset \mathbb{P}^{n-1}$ is arithmetically Cohen-Macaulay. On the other hand, for a nonsingular hyperplane section, we have the exact sequence

$$0 \to \mathcal{I}_S(-1) \to \mathcal{I}_S \to \mathcal{I}_{S \cap H \subset H} \to 0.$$

Using the associated long exact sequence, we deduce that $S \cap H$ is arithmetically Cohen-Macaulay. Repeating the same argument, we see that

any nonsingular linear section of S is arithmetically Cohen-Macaulay. The last claim then follows easily. $\square$

Now we can give some nonhomogeneous examples of special quadratic manifolds.

Proposition 2.11. *From Example 2.3, the Segre variety $S = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 = \mathbb{P}U$ is a special quadratic manifold associated with*

$$\sigma : \mathrm{Sym}^2 U \to W = H^0(\mathbb{P}^5, \mathcal{I}_S(2))^*$$

and $Z(\sigma)$ isomorphic to the Plücker embedding $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. A nonsingular linear section S' of S by a general subspace $U' \subset U$ of codimension $s \leq 2$ is a special quadratic manifold associated to a special system of quadrics

$$\sigma' : \mathrm{Sym}^2 U' \to W = H^0(\mathbb{P}^5, \mathcal{I}_S(2))^* = H^0(\mathbb{P}^{5-s}, \mathcal{I}_{S'}(2))^*$$

and $Z(\sigma')$ is equal to a nonsingular linear section of $\mathrm{Gr}(2,5)$ with codimension s.

Proof. Applying Proposition 2.6 repeatedly in combination with Corollary 2.8 and Proposition 2.9, we see that a general linear section S' of $\mathbb{P}^1 \times \mathbb{P}^2$ with codimension $s \leq 2$ is a special quadratic manifold associated to a special system of quadrics $\sigma' : \mathrm{Sym}^2 U' \to W = H^0(\mathbb{P}^5, \mathcal{I}_S(2))^*$ and $Z(\sigma')$ equal to a $\mathbb{P}^2$-general linear section of $\mathrm{Gr}(2,5)$ with codimension s. It is well-known (e.g. Remark 3.3.2 in [IP]) that all nonsingular sections of S with codimension $s \leq 2$ are projectively equivalent and all nonsingular sections of $\mathrm{Gr}(2,5)$ with codimension $s \leq 2$ are projectively equivalent. Thus we may say that $Z(\sigma')$ is any nonsingular linear section of $\mathrm{Gr}(2,5)$ of codimension $s \leq 2$. Using Lemma 2.10, we see $W^* = H^0(\mathbb{P}^{5-s}, \mathcal{I}_{S'}(2))$. $\square$

Proposition 2.12. *From Example 2.3, the Grassmannian $S = \mathrm{Gr}(2,5) \subset \mathbb{P}^9 = \mathbb{P}U$ is a special quadratic manifold associated with*

$$\sigma : \mathrm{Sym}^2 U \to W = H^0(\mathbb{P}^9, \mathcal{I}_S(2))^*$$

and $Z(\sigma)$ is isomorphic to the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$. A nonsingular linear section S' of S by a general subspace $U' \subset U$ of codimension $s \leq 3$ is a special quadratic manifold associated to a special system of quadrics

$$\sigma' : \mathrm{Sym}^2 U' \to W = H^0(\mathbb{P}^9, \mathcal{I}_S(2))^* = H^0(\mathbb{P}^{9-s}, \mathcal{I}_{S'}(2))^*$$

and $Z(\sigma')$ equals to a $\mathbb{P}^4$-general linear section of $Z(\sigma) = \mathbb{S}_5$ with codimension $s \leq 3$.

Proof. As in the proof of Proposition 2.11, this follows from Proposition 2.6, Corollary 2.8, Proposition 2.9 and Lemma 2.10, modulo the fact (see Remark 3.3.2 in [IP]) that all nonsingular linear sections of $\mathrm{Gr}(2,5)$ of a fixed codimension $s \leq 3$ are projectively equivalent. $\square$

Remark 2.13. (1) Let $\mathbb{S}_5 = S \subset \mathbb{P}U, \dim U = 16$, be the 10-dimensional Spinor variety. It is well-known that the dual variety $S^* \subset \mathbb{P}U^*$ is isomorphic to $S \subset \mathbb{P}U$ and the automorphism group $\mathrm{Aut}(S)$ acts transitively on S and $\mathbb{P}U \setminus S$. This implies that all nonsingular hyperplane sections of $\mathbb{S}_5$ are projectively equivalent.

(2) All nonsingular linear sections of $\mathbb{S}_5$ with codimension 2 contain a linear $\mathbb{P}^4$. One way to see this is using the fact that a general hyperplane section S_1 of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is isomorphic to a horospherical Fano manifold of Picard number 1, the case 4 in Theorem 1.7 of [P] (this fact follows from Mukai's classification [M]). From [P], the automorphism group $\mathrm{Aut}(S_1)$ has two orbits, an open orbit and a closed orbit, say $Q \subset S_1$, which is isomorphic to the 6-dimensional hyperquadric $\mathbb{Q}^6$. Let $\pi : \mathrm{Bl}_Q(S_1) \to S_1$ be the blow-up of S_1 along Q and let $E \subset \mathrm{Bl}_Q(S_1)$ be the exceptional divisor. Then by the proof of Lemma 1.17 [P], there exists a morphism $q : \mathrm{Bl}_Q(S_1) \to \mathbb{Q}^5$ which is a $\mathbb{P}^4$-bundle. The fibers of q are mapped to linear $\mathbb{P}^4$'s contained in S_1 and any linear $\mathbb{P}^4$ in S_1 arises this way. The intersection of $\mathbb{P}^4$ in S_1 with Q is a 3-dimensional hyperquadric $\mathbb{Q}^3 \subset \mathbb{P}^4$. Now suppose that a general nonsingular hyperplane section $H \subset S_1$ does not contain any linear $\mathbb{P}^4$. Then its proper image $H' \subset \mathrm{Bl}_Q(S_1)$ is a $\mathbb{P}^3$-bundle over $\mathbb{Q}^5$ and the exceptional divisor $E \cap H'$ is a $\mathbb{Q}^2$-bundle over $\mathbb{Q}^5$. On the other hand, the exceptional divisor $E \cap H'$ is a $\mathbb{P}^2$-bundle over the blow-up center $Q \cap H$, which is isomorphic to $\mathbb{Q}^5$. This leads to a contradiction by considering the Euler characteristic of $E \cap H'$. Thus any nonsingular hyperplane section of S_1 must contain a linear $\mathbb{P}^4$. (As a matter of fact, one can show that all nonsingular linear sections of $\mathbb{S}_5$ with codimension 2 are projectively equivalent. But as this is not directly related to our main results, we will not discuss its proof, which is more involved.)

(3) A general linear section S_3 of $\mathbb{S}_5$ with codimension 3 does *not* contain a linear $\mathbb{P}^4$. To see this, use Proposition 2.19 of Chapter III in [Z] which says that the set of linear $\mathbb{P}^4$'s on $\mathbb{S}_5 = S \subset \mathbb{P}U$ is parametrized by the dual variety S^*. This gives an embedding $S^* \subset \mathrm{Gr}(5, U)$. Let $U' \subset U$ be a general linear subspace of codimension $c > 0$ and regard $\mathrm{Gr}(5, U')$ as a submanifold of $\mathrm{Gr}(5, U)$ in a natural way. Then $S \cap \mathbb{P}U'$ contains a linear $\mathbb{P}^4$ if and only if $\mathrm{Gr}(5, U') \cap S^* \neq \emptyset$. By Theorem 10.8 of [Ha], the intersection $\mathrm{Gr}(5, U') \cap S^*$ must be either empty or of

dimension

$$\dim \mathrm{Gr}(5, U') + \dim S^* - \dim \mathrm{Gr}(5, U) = (50 - 5c) + 10 - 50 = 10 - 5c.$$

Thus the intersection is empty if $c > 2$. This also shows that, when $c = 2$, the intersection $\mathrm{Gr}(5, U') \cap S^*$, which is nonempty from (2), is finite, i.e., a general linear section S_2 in (2) contains finitely many linear $\mathbb{P}^4$'s.

The following result from Corollary 3.21 in [AS] is a converse to Proposition 2.11 and Proposition 2.12.

Proposition 2.14. *A nonsingular linear section of the homogeneous special quadratic manifolds in Example 2.3 is a special quadratic manifold if and only if it is a hyperquadric or one provided by Proposition 2.11 and Proposition 2.12.*

Our aim is to show that Example 2.3, Example 2.4, Proposition 2.11 and Proposition 2.12 exhaust all special quadratic manifolds. More precisely, we have the following classification, which gives a complete classification of special birational transformations of type (2,1).

Theorem 2.15. *A special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:*

(a) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ *for* $d \geq 1$;
(b) $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ *for* $d \geq 3$;
(c) *the 6-dimensional Grassmannian* $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$;
(d) *the 10-dimensional Spinor variety* $\mathbb{S}_5 \subset \mathbb{P}^{15}$;
(e) *a nonsingular codimension* ≤ 2 *linear section of* $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
(f) *a nonsingular codimension* ≤ 3 *linear section of* $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$.

The birational map ψ° is given by $H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$ except for (b) with $d \geq 6$, where ψ° can be given by some proper subspaces of $H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$ as in Example 2.4.

The proof of Theorem 2.15 will be given in Section 6.

3. Classification of $Y(\sigma)$

In this section, we classify varieties that can occur as the ψ-image $Y(\sigma)$ of a special quadratic manifold $S \subset \mathbb{P}U$. To simplify the notation, we will write Y for $Y(\sigma)$ and Z for $Z(\sigma)$. We will use the notation of Definition 2.1.

Proposition 3.1. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold. Then Y is nonsingular.*

Proof. Consider the $\mathbb{C}^*$-action on $\mathbb{P}(T\oplus U)$ given by $\lambda\cdot[t:u]=[t:\lambda u]$ for all $\lambda\in\mathbb{C}^*$. The fixed locus of this $\mathbb{C}^*$-action has two components: the isolated point $\mathbb{P}T$ and the hyperplane $\mathbb{P}U$. As S is contained in the fixed locus, this $\mathbb{C}^*$-action lifts to $\mathrm{Bl}_S(\mathbb{P}(T\oplus U))$. Since the morphism $\phi:\mathrm{Bl}_S(\mathbb{P}(T\oplus U))\to Z$ has connected fibers by Zariski main theorem, the $\mathbb{C}^*$-action on $\mathrm{Bl}_S(\mathbb{P}(T\oplus U))$ descends to a $\mathbb{C}^*$-action on Z such that ϕ is $\mathbb{C}^*$-equivariant. This implies that the proper image of $\mathbb{P}U$ under ϕ^o, which is Y, is contained in the fixed locus of the $\mathbb{C}^*$-action on Z. On the other hand, the $\mathbb{C}^*$-orbit of a general point of Z has a limit point in Y, hence Y is an irreducible component of the fixed locus of the $\mathbb{C}^*$-action on Z. This shows that Y is nonsingular because Z is nonsingular. $\square$

Definition 3.2. For a projective submanifold $M\subset\mathbb{P}^N$ and two distinct points $x\neq y\in M$, denote by $\ell_{x,y}\subset\mathbb{P}^N$ the line joining x and y. Such a line is called a *secant line* of M. The *secant variety* of M is

$$\mathrm{Sec}(M)=\text{ the closure of }\cup_{x\neq y\in M}\ell_{x,y}.$$

The *secant defect* of M is defined by $\delta_M=2\dim M+1-\dim\mathrm{Sec}(M)$.

Proposition 3.3. *Let $S\subset\mathbb{P}U$ be a special quadratic manifold. Let $\pi:\mathrm{Bl}_S(\mathbb{P}U)\to\mathbb{P}U$ be the blow-up morphism. Then* $\mathrm{Sec}(S)=\mathbb{P}U$ *and the morphism $\psi:\mathrm{Bl}_S(\mathbb{P}U)\to Y$ has the following properties:*

1. *ψ contracts the proper transform $\pi^{-1}[\ell]$ of a secant line ℓ of S not contained in S to a point in Y;*
2. *the fibers of the exceptional divisor $\pi|_F:F\to S$ are sent isomorphically to linear subspaces in Y; and*
3. *$\psi(F)=Y$.*

Proof. The equality $\mathrm{Sec}(S)=\mathbb{P}U$ is from Proposition 2.3(a) of [ES].

Recall that $\psi^o:\mathbb{P}U\dashrightarrow Y$ is induced by the projection $\sigma:\mathrm{Sym}^2U\to W$ composed with the second Veronese embedding $v_2:\mathbb{P}U\to\mathbb{P}\,\mathrm{Sym}^2U$. Let $\ell\subset\mathbb{P}U$ be a secant line of S not contained in S. Let $L\subset\mathrm{Sym}^2U$ be the 3-dimensional subspace spanned by $v_2(\ell)$. Since $v_2(\ell)$ intersects $\mathbb{P}\mathrm{Ker}(\sigma)$ at two distinct points, we have $\dim L\cap\mathrm{Ker}(\sigma)=2$. Thus $\sigma(L)$ is 1-dimensional, i.e., ψ contracts ℓ, proving (1).

Note that ψ can be viewed as the restriction of the natural projection $p:\mathrm{Bl}_{\mathbb{P}\mathrm{Ker}(\sigma)}(\mathbb{P}\,\mathrm{Sym}^2U)\to\mathbb{P}W$ to $\mathrm{Bl}_S(v_2(\mathbb{P}U))$. As the normal bundle of $\mathbb{P}\mathrm{Ker}(\sigma)$ in $\mathbb{P}\,\mathrm{Sym}^2U$ is $\mathcal{O}(1)^s$ for some s, the exceptional divisor of the blow-up is a product of two projective spaces, which can only be contracted by the projection to the other factor. Hence the projection p sends a fiber of the blow-up isomorphically to a linear subspace in $\mathbb{P}W$. This implies (2).

Finally, (3) is a consequence of $\mathrm{Sec}(S)=\mathbb{P}U$ and (1). $\square$

Let us recall the following basic terminology.

Definition 3.4. Let $M \subset \mathbb{P}^N$ be a projective submanifold.

(1) M is *conic-connected* if there exists an irreducible conic curve through two general points of M.
(2) M is a *prime Fano manifold* if $\mathrm{Pic}(M)$ is generated by $\mathcal{O}_M(1)$ and M is covered by lines.

Proposition 3.5. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold. Then Y is a conic-connected prime Fano manifold.*

Proof. Recall that $\psi^o : \mathbb{P}U \dashrightarrow Y$ is induced by the projection $\sigma : \mathrm{Sym}^2 U \to W$ composed with the second Veronese embedding $v_2 : \mathbb{P}U \to \mathbb{P}\,\mathrm{Sym}^2 U$. Since the Veronese variety $v_2(\mathbb{P}U) \subset \mathbb{P}(\mathrm{Sym}^2 U)$ is conic-connected, the image Y must be conic-connected.

Since $\mathrm{Bl}_S(\mathbb{P}U)$ has Picard number 2 and $\psi : \mathrm{Bl}_S(\mathbb{P}U) \to Y$ contracts some curves by Proposition 3.3 (1), we see that Y has Picard number 1. By Proposition 3.3 (2), lines cover Y. Thus Y is a prime Fano manifold. ☐

Proposition 3.6. *The secant defect $\delta = \delta_S$ of a special quadratic manifold $S \subset \mathbb{P}U$ satisfies $\delta = 2\dim S + 2 - \dim U$ and*

$$\dim Y = 2(\dim S - \delta) = 2(\dim U - \dim S - 2).$$

Proof. The equality $\delta = 2\dim S + 2 - \dim U$ is a consequence of $\mathrm{Sec}(S) = \mathbb{P}U$ in Proposition 3.3 and the equality $\dim Y = 2(\dim S - \delta)$ is from Corollary 2.3 of [AS]. Finally, $\dim S - \delta = \dim U - \dim S - 2$ from $\delta = 2\dim S + 2 - \dim U$. ☐

Proposition 3.7. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with secant defect δ. Then there is a natural identification $\mathrm{Bl}_S(\mathbb{P}(T \oplus U)) = \mathrm{Bl}_Y(Z)$ such that $\phi : \mathrm{Bl}_S(\mathbb{P}(T \oplus U)) \to Z$ coincides with the blow-up of Z along Y with the exceptional divisor $\mathrm{Bl}_S(\mathbb{P}U)$. In particular, the morphism $\psi : \mathrm{Bl}_S(\mathbb{P}U) \to Y$ is a $\mathbb{P}^{\delta+1}$-bundle.*

Proof. Since both Y and Z are nonsingular, Proposition 3.3 and Theorem 1.1 of [ES] imply that $\mathrm{Bl}_S(\mathbb{P}(T \oplus U))$ is also the blow-up of Z along Y. Thus ψ is a $\mathbb{P}^k$-bundle, where $k = \dim Z - \dim Y - 1$. By Proposition 3.6, we have $k = \delta + 1$. ☐

The following two propositions are direct consequences of Proposition 3.7.

Proposition 3.8. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with secant defect δ. A general fiber of the $\mathbb{P}^{\delta+1}$-bundle $\psi : \mathrm{Bl}_S(\mathbb{P}U) \to Y$ in Proposition 3.7 is sent to a linear subspace in $\mathbb{P}U$ by the blow-up morphism $\mathrm{Bl}_S(\mathbb{P}U) \to \mathbb{P}U$.*

Proof. Let $z \in \mathrm{Sec}(S) = \mathbb{P}U$ be a general point. Let $C_z(S) \subset \mathbb{P}U$ be the union of secant lines of S passing through z. By Proposition 2.3 (b) of [ES], the cone $C_z(S) \subset \mathbb{P}U$ is a linear subspace of dimension $\delta + 1$. But by Proposition 3.3 (1), the cone $C_z(S)$ is contained in the image of a fiber of ψ. Thus $C_z(S)$ must coincide with the image of a fiber of ψ. □

Proposition 3.9. *For a special quadratic manifold S, set $n = \dim U$ and $d = \dim S$. Then the Euler numbers of S and Y are related by*

$$\chi(S) = \frac{(\delta+2)\chi(Y) - n}{n-d-2}.$$

Proof. Since the exceptional divisor F of the blow-up $\mathrm{Bl}_S(\mathbb{P}^{n-1}) \to \mathbb{P}^{n-1}$ is a $\mathbb{P}^{n-d-2}$-bundle over S, we have

$$\chi(\mathbb{P}^{n-1}) - \chi(S) = \chi(\mathrm{Bl}_S(\mathbb{P}^{n-1})) - \chi(F) = \chi(\mathrm{Bl}_S(\mathbb{P}^{n-1})) - (n-d-1)\chi(S),$$

which gives $\chi(\mathrm{Bl}_S(\mathbb{P}^{n-1})) = n + (n-d-2)\chi(S)$. On the other hand, the map $\mathrm{Bl}_S(\mathbb{P}^{n-1}) \to Y$ is a $\mathbb{P}^{\delta+1}$-bundle from Proposition 3.7, which implies $\chi(\mathrm{Bl}_S(\mathbb{P}^{n-1})) = (\delta+2)\chi(Y)$. Combining the two gives the desired equality. □

We are ready to have a classification of Y.

Theorem 3.10. *Let $S \subset \mathbb{P}U$ be a special quadric manifold and let $c = \dim U - \dim S - 1$ be its codimension. Recall that $\dim Y = 2(c-1)$ from Proposition 3.6. Then $Y \subset \mathbb{P}W$ is isomorphic to one of the following:*

- (Y1) $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}W$;
- (Y2) *a nonsingular quadric hypersurface* $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}W \cong \mathbb{P}^{2c-1}$; *or*
- (Y3) *a biregular projection of the Plücker embedding* $\mathrm{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$.

Proof. By Proposition 3.3, the (nondegenerate) projective submanifold $Y \subset \mathbb{P}W$ of dimension $2(c-1)$ is covered by linear subspaces of dimension $c-1$. Thus we can use Eichi Sato's classification, Main Theorem in [S], of nonsingular projective varieties of dimension $\leq 2(c-1)$ covered by linear subspaces of dimension $(c-1)$. Since Y has Picard number 1 from Proposition 3.5, Sato's classification shows that $Y \subset \mathbb{P}W$ must be one of the three listed varieties. □

We have the following topological consequence.

Corollary 3.11. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with $d = \dim S$ and $n = \dim U$. Then all odd Betti numbers of S vanish. In particular, the Euler number of S satisfies $\chi(S) \geq d + 1$.*

Proof. As the morphism $\mathrm{Bl}_S(\mathbb{P}^{n-1}) \to Y$ is a $\mathbb{P}^{\delta+1}$-bundle by Proposition 3.7 and odd Betti numbers of Y vanish from Theorem 3.10, the odd Betti numbers of $\mathrm{Bl}_S(\mathbb{P}^{n-1})$ vanish. This implies that odd Betti numbers of S vanish by the formulae for blow-ups in Chapter 4, Section 6, p605 of [GH]. □

In the case of (Y2) of Theorem 3.10, we have the following topological consequence.

Proposition 3.12. *For a special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ with codimension c and $n \geq 3$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then either $\delta \geq \frac{d}{2}$ or $\delta = \frac{d}{2} - 1$.*

Proof. Putting $\chi(\mathbb{Q}^{2(c-1)}) = 2c$ in Proposition 3.9, we obtain

$$\chi(S) = 2\delta + \frac{n}{n-d-2}. \tag{3.1}$$

Set $k := \dfrac{n}{n-d-2}$. The above equation implies that k is a natural number, hence $2 \leq k \leq n$.

Assume that $\delta < \frac{d}{2}$, i.e. $2d + 2 - n < \frac{d}{2}$, which is equivalent to $3d + 5 \leq 2n$, implying $d \leq \dfrac{2n-5}{3}$ and

$$n = k(n-d-2) \geq k(n - \frac{2n-5}{3} - 2) = \frac{k(n-1)}{3}.$$

Let us check case-by-case the possible values of k.

(1) When $4 \leq k \leq n$. Then $3n \geq 4(n-1)$ gives $n = 4, d = 1, \delta = 0$ and $\chi(S) = 4$. But there is no smooth curve with $\chi(S) = 4$, a contradiction.
(2) When $k = 3$. Then $2n = 3d + 6$ and $\delta = \frac{d}{2} - 1$.
(3) When $k = 2$. We have $n = 2(n-d-2)$, which yields $n = 2d+4$, a contradiction to $\mathrm{Sec}(S) = \mathbb{P}^{n-1}$ of Proposition 3.3.

We conclude that $\delta = \frac{d}{2} - 1$ is the only possibility. □

4. Review of results on QEL-manifolds

For the proof of Theorem 2.15, we will use the notion of QEL-manifolds introduced by Russo in [R2]. In this section, we review some results on QEL-manifolds from [IR2] and [R2] that are needed for our purpose.

Definition 4.1. Let $S \subset \mathbb{P}V$ be a nondegenerate submanifold with secant defect δ.

(1) For a point $z \in \mathrm{Sec}(S) \setminus S$, the *entry locus* of S with respect to z is the subvariety $\Sigma_z(S) \subset S$ defined by

$$\Sigma_z(S) = \text{ the closure of } \{x \in S \ \mid \ z \in \ell_{x,y} \text{ for some } y \in S, y \neq x\}.$$

The cone over $\Sigma_z(S)$ with the vertex at z is denoted by $C_z(S)$. If z is general, then $\Sigma_z(S) = S \cap C_z(S)$.

(2) S is said to be a *quadratic entry locus manifold* (QEL-manifold in abbreviation) if for a general $z \in \mathrm{Sec}(S)$, the cone $C_z(S) \subset \mathbb{P}V$ is a linear subspace of dimension $\delta + 1$ and the entry locus $\Sigma_z(S) \subset C_z(S)$ is a nonsingular quadratic hypersurface in the linear subspace.

This notion is relevant to us by the following.

Proposition 4.2. *A special quadratic manifold $S \subset \mathbb{P}U$ is a QEL-manifold and is linearly normal.*

Proof. Proposition 2.3 (b) of [ES] says that a special quadratic manifold S is a QEL-manifold. Proposition 1.3 of [R2] says that a QEL-manifold $S \subset \mathbb{P}U$ is linearly normal if $\mathrm{Sec}(S) = \mathbb{P}U$. But the latter condition holds for a special quadratic manifold by Proposition 3.3. ☐

In what way is Proposition 4.2 useful to us? The biggest advantage of considering QEL-manifolds is that one can use an inductive argument via varieties of minimal rational tangents. Let us recall the definition.

Definition 4.3. Let $M \subset \mathbb{P}^N$ be a prime Fano manifold. The VMRT at a point x is the subvariety $\mathcal{C}_x \subset \mathbb{P}T_xM$ consisting of tangent directions to lines on M through x. When x is a general point of M, the VMRT at x is nonsingular.

Examples 4.4. An irreducible Hermitian symmetric space of compact type is a homogeneous space $M = G/P$ with a simple Lie group G and a maximal parabolic subgroup P such that the isotropy representation of P on $T_x(M)$ at a base point $x \in M$ is irreducible. The highest weight orbit of the isotropy action on $\mathbb{P}T_x(M)$ is exactly the VMRT at x. The following table (e.g. Section 3.1 [FH]) collects basic information on these varieties.

Type	I.H.S.S. M	VMRT S	$S \subset \mathbb{P}T_x(M)$
I	$\mathrm{Gr}(a, a+b)$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	Segre
II	$\mathbb{S}_n$	$\mathrm{Gr}(2,n)$	Plücker
III	$\mathrm{Lag}(2n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$\mathbb{Q}^n$	$\mathbb{Q}^{n-2}$	Hyperquadric
V	$\mathbb{OP}^2$	$\mathbb{S}_5$	Spinor
VI	$E_7/(E_6 \times U(1))$	$\mathbb{OP}^2$	Severi

Examples 4.5. Let Σ be an n-dimensional vector space endowed with a skew-symmetric 2-form ω of maximal rank. The symplectic Grassmannian $M = \mathrm{Gr}_\omega(k, \Sigma)$ is the variety of all k-dimensional isotropic subspaces of Σ, which is not homogeneous if n is odd. Let W and Q be vector spaces of dimensions $k \geq 2$ and m respectively. Let $\mathbf{t}$ be the tautological line bundle over $\mathbb{P}W$. The VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ of $\mathrm{Gr}_\omega(k, \Sigma)$ at a general point is isomorphic to the projective bundle $\mathbb{P}((Q \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$ over $\mathbb{P}W$ with the projective embedding given by the complete linear system

$$H^0(\mathbb{P}W, (Q \otimes \mathbf{t}^*) \oplus (\mathbf{t}^*)^{\otimes 2}) = (W \otimes Q)^* \oplus \mathrm{Sym}^2 W^*.$$

An inductive argument in studying QEL-manifolds is provided by the following result from Theorem 2.1, Theorem 2.3 and Theorem 2.8 in [R2].

Proposition 4.6 ([R2])**.** *Let $S^d \subset \mathbb{P}U$ be a d-dimensional QEL-manifold with secant defect δ.*

(1) *If $\delta \geq 1$ and S is a prime Fano manifold, then $d + \delta$ is even and $K_S^{-1} = \mathcal{O}(\frac{d+\delta}{2})$.*
(2) *If $\delta \geq 3$, then $d - \delta$ is divisible by $2^{[\frac{\delta-1}{2}]}$, the QEL-manifold S is a prime Fano manifold and the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at a general point is itself a QEL-manifold with $\dim \mathcal{C}_x = \frac{d+\delta-4}{2}$ and $\delta_{\mathcal{C}_x} = \delta - 2$. In fact, points of $\mathcal{C}_x$ corresponding to the lines through x contained in an entry locus of S through x form an entry locus of $\mathcal{C}_x$.*

Since in many cases the VMRT $\mathcal{C}_x$ at a general $x \in S$ gives a good deal of information for S, if a QEL-manifold S has $\delta \geq 3$, Proposition 4.6 enables us to reduce a problem on S to a lower-dimensional QEL-manifold $\mathcal{C}_x$. Notice that such an inductive argument is not available for special quadratic manifolds. This is why Proposition 4.2 is useful to us. The inductive argument for QEL-manifolds works perfectly when $\delta_S \geq \frac{1}{2} \dim S$ and we have the following two classification results due to Russo.

Proposition 4.7 (Corollary 3.1 in [R2])**.** *Let $S^d \subset \mathbb{P}^N$ be a d-dimensional QEL-manifold with $\frac{d}{2} < \delta < d$. Then $S \subset \mathbb{P}^N$ is isomorphic to one of the followings.*

(A1) *the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;*
(A2) *the Plücker embedding $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$;*
(A3) *the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;*
(A4) *a general hyperplane section of the Plücker embedding $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$;*

(A5) *a general hyperplane section of the Spinor variety* $\mathbb{S}_5 \subset \mathbb{P}^{15}$.

Proposition 4.8 (Corollary 3.2 in [R2])**.** *Let* $S^d \subset \mathbb{P}^N$ *be a* d*-dimensional QEL-manifold with* $\frac{d}{2} = \delta$. *Then* $d = 2, 4, 8$ *or* 16 *and* $S \subset \mathbb{P}^N$ *is isomorphic to one of the followings.*

(B1) *a general hyperplane section of the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$*;*
(B2) *the Veronese surface* $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$*;*
(B3) *the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$*;*
(B4) *the Segre variety* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$*;*
(B5) *a general codimension-2 linear section of the Plücker embedding* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$*;*
(B6) *a general codimension-2 linear section of the Spinor variety* $\mathbb{S}_5 \subset \mathbb{P}^{15}$*;*
(B7) *the Plücker embedding* $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$*;*
(B8) *the* E_6*-Severi variety* $\mathbb{OP}^2 \subset \mathbb{P}^{26}$.

Remark 4.9. Corollary 3.2 of [R2] lists one hypothetical case of $d = 16$ in addition to (B8). That case does not exist in view of Main Theorem of Section 2 in [Mo]. Corollary 3.2 of [R2] is stated for "LQEL-manifolds", a weaker notion than QEL-manifolds, with $\frac{d}{2} = \delta$. Because of this, the list there includes some biregular projections of QEL-manifolds. Since biregular projections of QEL-manifolds can not be QEL-manifolds by Proposition 1.3 of [R2], we have the above list.

Note that the inductive argument using Proposition 4.6 cannot be continued if $\delta \leq 2$. This is a major difficulty in the study of QEL-manifolds. In our case, however, thanks to the restriction coming from Theorem 3.10, such low defect cases can be handled by a number of explicit classification results on QEL-manifolds in the following two extreme cases.

Proposition 4.10. *Let* $S^d \subset \mathbb{P}^N$ *be a QEL-manifold with* $\mathrm{Sec}(S) = \mathbb{P}^N$ *and* $\delta \geq 2$. *If* S *is a prime Fano manifold with* $K_S^{-1} = \mathcal{O}(d-2)$*, then it is projectively equivalent to one of the following:*

(M1) *the 10-dimensional Spinor variety* $\mathbb{S}_5 \subset \mathbb{P}^{15}$*;*
(M2) *a nonsingular linear section of* $\mathbb{S}_5 \subset \mathbb{P}^{15}$ *of codimension* ≤ 4.

Proof. The assumption $\mathrm{Sec}(S) = \mathbb{P}^N$ implies that $S \subset \mathbb{P}^N$ is linearly normal by Proposition 1.3 of [R2]. By Proposition 4.6, we have $d + \delta = 2(d-2)$, hence $\delta = d - 4$ and $N = 2d + 1 - \delta = d + 5$. By our assumption $\delta \geq 2$, we obtain $d \geq 6$.

We will use Mukai's classification in [M] of linearly normal prime Fano manifolds $S^d \subset \mathbb{P}^N$ with $K_S^{-1} = \mathcal{O}(d-2)$. Mukai's classification is in terms of the genus $g := \frac{1}{2}\deg(S) + 1$. If $g \leq 5$, then such S is

a complete intersection by Remark 2 in [M]. Since a QEL-manifold that is a complete intersection must be a hyperquadric by Proposition 3.4 of [IR2], we have only $g \geq 6$. The classification for $g \geq 6$ is listed in Example 5.2.2 and Theorem 5.2.3 of [IP]. In the list, the only possibilities of $S^d \subset \mathbb{P}^{d+5}$ with $d \geq 6$ occur in Example 5.2.2 (vi) and these are exactly (M1) and (M2). $\square$

For the next classification result, we need the notion of rational normal scrolls.

Definition 4.11. For integers $a_1, \cdots, a_n \geq 1$, the *rational normal scroll* $S(a_1, \cdots, a_n)$ is the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ over $\mathbb{P}^1$, embedded into $\mathbb{P}^N$ with $N = \sum_i (a_i + 1) - 1$ such that the fiber of the projective bundle is mapped to linear subspaces in $\mathbb{P}^N$.

The following is proved in Theorem 3 of [R1].

Proposition 4.12. *Let $S \subset \mathbb{P}^{2d+1}$ be a QEL-manifold with $\delta = 0$.*

(1) *When* $\dim S = 1$*, the curve $S \subset \mathbb{P}^3$ is isomorphic to the twisted cubic.*
(2) *When* $\dim S = 2$*, there are three possibilities for the surface $S \subset \mathbb{P}^5$:*
  (a) *a Del Pezzo surface of degree 5 in $\mathbb{P}^5$;*
  (b) *the rational normal scroll* $S(2, 2)$*;*
  (c) *the rational normal scroll* $S(1, 3)$.

5. Intersection of entry loci on QEL-manifolds

In this section, we study the intersection of entry loci on QEL-manifolds satisfying certain additional conditions. Although our main purpose is to use this in the proof of Theorem 2.15, our results here have independent interest in projective geometry of QEL-manifolds. The following notation will be used.

Notation 5.1. For any subset $A \subset \mathbb{P}^{n-1}$, its linear span will be denoted by $\langle A \rangle \subset \mathbb{P}^{n-1}$. For a point $s \in \mathbb{P}^{n-1}$, we will write $\langle A, s \rangle$ in place of $\langle A \cup \{s\} \rangle$. For a projective variety $S \subset \mathbb{P}^{n-1}$ covered by lines and a point $s \in S$, we denote by $\mathrm{Locus}(s) \subset S$ the locus of lines on S passing through s. Thus for a general point $x \in S$,

$$\dim \mathrm{Locus}(x) = \dim \mathcal{C}_x + 1.$$

If $s \in S$ is a nonsingular point, then $\mathbf{T}_s S \subset \mathbb{P}^{n-1}$ denotes the projective tangent space of S at s. When $S \subset \mathbb{P}^{n-1}$ is a QEL-manifold with $\delta > 0$, for a point $u \in \mathrm{Sec}(S)$, we recall from Definition 4.1 that $\Sigma_u(S)$ is the entry locus of S and $C_u(S) = \langle \Sigma_u(S) \rangle$.

Condition 5.2. We consider the following condition on S:

For a general $u \in \operatorname{Sec}(S)$, we have $\Sigma_t(S) = \Sigma_u(S)$ for all $t \in \langle \Sigma_u \rangle \setminus \Sigma_u$.

Condition 5.2 is relevant to us by the following fact from Remark 2.4 [AS].

Lemma 5.3. *A special quadratic manifold satisfies Condition 5.2.*

Proposition 5.4. *Let $S \subsetneq \mathbb{P}^{n-1}$ be a QEL-manifold with $\delta > 0$ satisfying Condition 5.2. Fix a general entry locus $\Sigma = \Sigma_z(S)$ by choosing a general point $z \in \operatorname{Sec}(S)$ and fix a general point $x \in \Sigma$. There exists a Zariski open subset $O \subset S$ with $O \cap \Sigma = \emptyset$ such that for any $s \in O$, we have the following.*

1. *Either* $\operatorname{Locus}(s) \cap \Sigma = \emptyset$ *for all* $s \in O$ *or* $\operatorname{Locus}(s) \cap \Sigma \nsubseteq \mathbf{T}_x\Sigma \cap \Sigma$ *for all* $s \in O$.
2. *There exists a unique entry locus of S passing through x and s, to be denoted by $\Sigma_s \in S$, such that $\Sigma_s \subset \langle \Sigma_s \rangle$ is a smooth hyperquadric.*
3. $\langle \mathbf{T}_x S, s \rangle \cap S = (\mathbf{T}_x S \cap S) \cup \Sigma_s$.
4. *The line $\ell_{s,x}$ is not contained in S.*
5. *For any $t \in \langle \Sigma_s \rangle \setminus \Sigma_s$, we have $C_t(S) = \langle \Sigma_s \rangle$ and $\Sigma_t(S) = \Sigma_s$.*
6. *Define $P_s := \Sigma_s \cap \Sigma$. Then P_s coincides with $\langle \Sigma_s \rangle \cap \langle \Sigma \rangle$ and is a linear subspace of $\mathbb{P}^{n-1}$. In particular, we have $P_s \subset \mathbf{T}_x\Sigma \cap \Sigma$.*
7. *Let $p_\Sigma : S \setminus \Sigma \to \mathbb{P}^{n-\delta-3}$ be the projection from $\langle \Sigma \rangle$. Then p_Σ is a smooth morphism at s.*

We will use the following lemma.

Lemma 5.5 ([IR2], Lemma 1.2 and Theorem 2.3)**.** *Let $S^d \subset \mathbb{P}^{n-1}$ be a QEL-manifold with $\delta > 0$. Then*

(i) Through two general points of S, there passes a unique entry locus, which is a smooth hyperquadric in its linear span.

(ii) Let $x \in S$ be a general point and let $p_x : S \dashrightarrow \mathbb{P}^{n-d-2}$ be the tangential projection from $\mathbf{T}_x S$. Then the closure of the fiber of p_x through a general point $s \in S$ is the entry locus passing through s and x, hence it is a smooth hyperquadric of dimension δ.

Proof of Proposition 5.4. For each of the conditions (1)-(7), we will show that there exists a Zariski open subset of S every element of which satisfies the condition. Then O can be taken as the intersection of these open subsets.

For (1), assume that $\operatorname{Locus}(s) \cap \Sigma$ is non-empty for a general $s \in S$. Then

$$\bigcup_{y \in \Sigma \setminus \mathbf{T}_x\Sigma} \operatorname{Locus}(y)$$

covers an open subset of S and for any s inside this open subset,

$$\mathrm{Locus}(s) \cap \Sigma \not\subseteq \mathbf{T}_x\Sigma \cap \Sigma.$$

(2) follows from Lemma 5.5 (i) while (3) follows from Lemma 5.5 (ii).

Note that we have $\mathrm{Locus}(x) \neq S$ as $S \neq \mathbb{P}^{n-1}$. Thus any point $s \in S \setminus \mathrm{Locus}(x)$ satisfies the condition (4).

(5) is clear from Condition 5.2.

For (6), we take the open subset in (5). If $\langle \Sigma_s \rangle \cap \langle \Sigma \rangle$ contains a point not on S, say z, then $\langle \Sigma_s \rangle = C_z = \langle \Sigma \rangle$ by (5), which gives a contradiction. Thus $\langle \Sigma_s \rangle \cap \langle \Sigma \rangle \subset S$, hence $\Sigma_s \cap \Sigma = \langle \Sigma_s \rangle \cap \langle \Sigma \rangle$ is a linear subspace. The inclusion $P_s \subset \mathbf{T}_x\Sigma \cap \Sigma$ follows from the fact that $P_s \subset \Sigma$ is linear and $x \in P_s$.

(7) is obviously true for a general point s. □

Proposition 5.6. *In Proposition 5.4, there exists a linear subspace $W_s \subset \mathbb{P}^{n-1}$ for each $s \in O$ such that*

$$s \in W_s,\ \dim W_s = \dim P_s,\ \langle P_s \cap W_s, s\rangle = W_s \text{ and } \langle P_s, s\rangle \cap \Sigma_s = P_s \cup W_s.$$

In particular, $\langle P_s, s\rangle \not\subseteq \Sigma_s$ for each $s \in O$.

Proof. As the line $\ell_{s,x}$ is not contained in S by Proposition 5.4 (4), we see that $\langle P_s, s\rangle \not\subseteq \Sigma_s$. As $\langle P_s, s\rangle$ is a linear subspace of $\langle \Sigma_s \rangle$ and Σ_s is a quadric hypersurface in $\langle \Sigma_s \rangle$, the intersection $\langle P_s, s\rangle \cap \Sigma_s$ is a hypersurface of degree ≤ 2 in $\langle P_s, s\rangle$, which contains already the linear subspace P_s. As $s \notin P_s$, there exists a linear subspace W_s with $\dim W_s = \dim P_s$ satisfying

$$s \in W_s \text{ and } \langle P_s, s\rangle \cap \Sigma_s = P_s \cup W_s.$$

Moreover $P_s \cap W_s$ must be a hyperplane in W_s disjoint from s. Thus $\langle P_s \cap W_s, s\rangle = W_s$. □

Proposition 5.7. *In Proposition 5.4, assume furthermore that $S \subset \mathbb{P}^{n-1}$ is defined by quadratic equations. Then for any $s \in O \subset S$, there exists a linear subspace $F_s \subset S$ such that*

$$\langle \Sigma, s\rangle \cap S = \Sigma \cup F_s. \tag{5.1}$$

Moreover, we have $F_s = \langle \mathrm{Locus}(s) \cap \Sigma, s\rangle$ and $\dim F_s = \dim(\mathrm{Locus}(s) \cap \Sigma) + 1$. In particular, $\dim F_s = 0$ if and only if $\mathrm{Locus}(s) \cap \Sigma = \emptyset$.

Proof. Let F_s be the closure of the fiber of p_Σ through s, which is nonsingular at s by Proposition 5.4 (7). We claim that F_s is a cone over $\mathrm{Locus}(s) \cap \Sigma$ with vertex s. Then F_s must be linear as it is nonsingular at s and Proposition 5.7 follows.

To prove the claim, pick any $s' \in (\langle \Sigma, s\rangle \cap S) \setminus \Sigma$ with $s \neq s'$. It suffices to show that $\ell_{s,s'} \subset S$, which implies

$$s' \in \langle \mathrm{Locus}(s) \cap \Sigma, s\rangle \subset \langle \Sigma, s\rangle \cap S,$$

proving the claim.

As $\langle s, \Sigma\rangle = \langle s', \Sigma\rangle$, the line $\ell_{s,s'}$ intersects $\langle \Sigma\rangle$ at a single point, say $t \in \langle \Sigma\rangle$. Suppose that $t \notin \Sigma$. Then $\Sigma_t(S) = \Sigma$ by Condition 5.2. As $\ell_{s,s'}$ passes through t, we have $s, s' \in \Sigma_t(S) = \Sigma$, a contradiction. Thus $t \in \Sigma$, which implies that the line $\ell_{s,s'}$ has 3 intersection points with S. From the assumption that S is defined by quadratic equations, we have $\ell_{s,s'} \subset S$, proving the claim. □

Proposition 5.8. *Assume that $S \subset \mathbb{P}^{n-1}$ is a QEL-manifold with $\delta > 0$ that is defined by quadrics and satisfies Condition 5.2. Let O be as in Proposition 5.4. Then for any $s \in O$, we have*

$$\langle \mathbf{T}_x\Sigma, s\rangle \cap S = (\mathbf{T}_x\Sigma \cap S) \cup W_s \tag{5.2}$$

where W_s is as in Proposition 5.6.

It is convenient to recall the following straightforward lemma.

Lemma 5.9. *Let $L_1, L_2 \subset \mathbb{P}^{n-1}$ be two linear subspaces. Then*
 (i) *$\langle L_1 \cap L_2, s\rangle = \langle L_1, s\rangle \cap L_2$ if $s \in L_2$; and*
 (ii) *$\langle L_1, s\rangle \cap L_2 = L_1$ if $L_1 \subset L_2$ and $s \notin L_2$.*

Proof of Proposition 5.8. To start with, we claim the following relation

$$\langle P_s, s\rangle \cap \Sigma_s \subset \langle \mathbf{T}_x\Sigma, s\rangle \cap \Sigma_s \subsetneq \langle \Sigma, s\rangle \cap \langle \Sigma_s\rangle = \langle P_s, s\rangle. \tag{5.3}$$

The first inclusion is from $P_s \subset \mathbf{T}_x\Sigma$ in Proposition 5.4 (6). The second inclusion is from $\mathbf{T}_x\Sigma \subset \langle \Sigma, s\rangle$. The last equality follows from

$$\langle P_s, s\rangle = \langle \langle \Sigma\rangle \cap \langle \Sigma_s\rangle, s\rangle = \langle \Sigma, s\rangle \cap \langle \Sigma_s\rangle$$

which is a consequence of Proposition 5.4 (6) and Lemma 5.9 (i). Finally, the inequality in the middle is by $\langle P_s, s\rangle \not\subset \Sigma_s$ from Proposition 5.6.

In (5.3), the proper subvariety $\langle \mathbf{T}_x\Sigma, s\rangle \cap \Sigma_s$ of $\langle P_s, s\rangle$ is a subvariety of degree ≤ 2. Since $\langle P_s, s\rangle$ is a linear subspace of $\langle \Sigma_s\rangle$ from Proposition 5.4 (6) and $\Sigma_s \subset \langle \Sigma_s\rangle$ is a quadric hypersurface from Proposition 5.4 (2), the intersection $\langle P_s, s\rangle \cap \Sigma_s$ must be a hypersurface of degree 2 in $\langle P_s, s\rangle$. It follows that

$$\langle P_s, s\rangle \cap \Sigma_s = \langle \mathbf{T}_x\Sigma, s\rangle \cap \Sigma_s. \tag{5.4}$$

From $\mathbf{T}_x\Sigma \subset \mathbf{T}_xS$, we have the tautological relation

$$\langle \mathbf{T}_x\Sigma, s\rangle \cap S = \langle \mathbf{T}_x\Sigma, s\rangle \cap (\langle \mathbf{T}_xS, s\rangle \cap S) \tag{5.5}$$

Combining (5.5) with $\langle \mathbf{T}_x S, s\rangle \cap S = \Sigma_s \cup (\mathbf{T}_x S \cap S)$ from Proposition 5.4 (3), we obtain

$$\langle \mathbf{T}_x \Sigma, s\rangle \cap S = (\langle \mathbf{T}_x \Sigma, s\rangle \cap \Sigma_s) \cup (\langle \mathbf{T}_x \Sigma, s\rangle \cap \mathbf{T}_x S \cap S). \tag{5.6}$$

For the first term on the righthand side of (5.6), we have

$$\langle \mathbf{T}_x \Sigma, s\rangle \cap \Sigma_s = \langle P_s, s\rangle \cap \Sigma_s = P_s \cup W_s \tag{5.7}$$

where the first equality is from (5.4) and the second equality is from Proposition 5.6. For the second term on the right hand side of (5.6), we have

$$\langle \mathbf{T}_x \Sigma, s\rangle \cap \mathbf{T}_x S \cap S = \mathbf{T}_x \Sigma \cap S \tag{5.8}$$

because $\langle T_x \Sigma, s\rangle \cap \mathbf{T}_x S = \mathbf{T}_x \Sigma$ by Lemma 5.9 (ii). Putting (5.7) and (5.8) into (5.6), we obtain

$$\langle \mathbf{T}_x \Sigma, s\rangle \cap S = P_s \cup W_s \cup (\mathbf{T}_x \Sigma \cap S).$$

Recalling $P_s \subset \mathbf{T}_x \Sigma \cap \Sigma$ from Proposition 5.4 (6), we have (5.2). □

Theorem 5.10. *In Proposition 5.8, assume that S is prime Fano and that* $\mathrm{Locus}(s) \cap \Sigma \neq \emptyset$ *for $s \in O$. Then*

(i) $\dim F_s = \frac{3\delta - d}{2}$;

(ii) *W_s is a codimension-1 linear subspace of F_s; and*

(iii) $\Sigma \cap \Sigma_s \simeq \mathbb{P}^{\frac{3\delta-d-2}{2}}$.

Proof. Consider the incidence variety

$$I = \text{ the closure of } \{(t, y) \in \Sigma \times O | L_{ty} \subset S\}$$

and the two projections $p_1 : I \to \Sigma$ and $p_2 : I \to S$. As S is a prime Fano QEL-manifold, Proposition 4.6 gives $\dim \mathcal{C}_t = \frac{d+\delta}{2} - 2$ for a general $t \in S$, which implies

$$\dim \mathrm{Locus}(t) = \dim \mathcal{C}_t + 1 = \frac{d+\delta}{2} - 1.$$

Since we have chosen our Σ generally, the dimension of a general fiber of p_1 is equal to $\dim \mathrm{Locus}(t)$ for general $t \in S$. Thus

$$\dim I - \dim \Sigma = \frac{d+\delta}{2} - 1 \text{ and } \dim I = \delta + \frac{d+\delta}{2} - 1.$$

The assumption $\mathrm{Locus}(s) \cap \Sigma \neq \emptyset$ for $s \in O$ implies that p_2 is surjective by Proposition 5.4 (1). Thus the dimension of a general fiber of p_2 is

$$\dim I - \dim S = \delta + \frac{d+\delta}{2} - 1 - d = \frac{3\delta - d}{2} - 1.$$

Since the fiber of p_2 over a general $s \in O$ is $\mathrm{Locus}(s) \cap \Sigma$ and $F_s = \langle \mathrm{Locus}(s) \cap \Sigma, s\rangle$ from Proposition 5.7, we have $\dim F_s = \frac{3\delta - d}{2}$, proving (i).

Comparing equations (5.1) and (5.2), we see that $W_s \subset F_s$ is a linear subspace of codimension ≤ 1. Suppose that $W_s = F_s$. Note that $P_s \cap W_s \subset \text{Locus}(s) \cap \Sigma$ from $P_s \subset \Sigma$ and $W_s \subset \text{Locus}(s)$. As $F_s = \langle s, \text{Locus}(s) \cap \Sigma \rangle$ from Proposition 5.7 and $W_s = \langle s, P_s \cap W_s \rangle$ from Proposition 5.6, the hypothesis $F_s = W_s$ implies $\text{Locus}(s) \cap \Sigma = P_s \cap W_s$. Combining this with Proposition 5.4 (6), we have

$$\text{Locus}(s) \cap \Sigma \subset P_s \subset \Sigma \cap \mathbf{T}_x \Sigma$$

which contradicts Proposition 5.4 (1). Thus W_s is of codimension one in F_s, proving (ii).

Finally, the equality $\dim P_s = \dim W_s$ in Proposition 5.6 implies

$$\dim(\Sigma \cap \Sigma_s) = \dim W_s = \dim F_s - 1 = \frac{3\delta - d - 2}{2},$$

proving (iii). □

We close this section with one explicit example of the intersection of entry loci. It is a simple consequence of the following elementary lemma, whose proof will be skipped.

Lemma 5.11. *Let V be a vector space of dimension $2m \geq 6$, equipped with a non-degenerate quadratic form $Q : V \times V \to \mathbb{C}$.*

(1) *The two components $\mathbb{S}^+(V; Q)$ and $\mathbb{S}^-(V; Q)$ of the space of m-dimensional Q-isotropic subspaces of V admit projective embeddings, realizing them as prime Fano manifolds biregular to each other.*

(2) *Let $W \subset V$ be a Q-isotropic subspace of dimension $\leq m - 3$ and let*

$$W^\perp := \{v \in V,\ Q(v, w) = 0 \text{ for all } w \in W.\}.$$

The quotient space $W^\perp/W$ with $\dim W^\perp/W \geq 6$ is equipped with a non-degenerate quadratic from Q^W induced by Q and we have natural inclusions

$$\mathbb{S}^+(W^\perp/W; Q^W) \subset \mathbb{S}^+(V; Q) \text{ and } \mathbb{S}^-(W^\perp/W; Q^W) \subset \mathbb{S}^-(V; Q)$$

which induce isomorphisms between their Picard groups.

(3) *When $W = \mathbb{C}v$ for some $0 \neq v \in V$, let us write $\mathbb{S}^+(W^\perp/W; Q^W)$ as $\mathbb{S}^v$. For two independent vectors $v, v' \in V$, if $Q(v, v') \neq 0$, then $\mathbb{S}^v \cap \mathbb{S}^{v'} = \emptyset$, while if $Q(v, v') = 0$, then*

$$\mathbb{S}^v \cap \mathbb{S}^{v'} = \mathbb{S}^+(W^\perp/W; Q^W) \text{ for } W = \mathbb{C}v + \mathbb{C}v'.$$

We denote by $\mathbb{S}_m$ the Spinor variety $\mathbb{S}^+(V; Q)$ of Lemma 5.11. We have the following consequence.

Proposition 5.12. *The intersection of any two distinct entry loci of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is either empty or $\mathbb{P}^3$.*

Proof. It is well-known that the embedding of $\mathbb{S}_m$ as a prime Fano manifold in Lemma 5.11 becomes an isomorphism $\mathbb{S}_3 \cong \mathbb{P}^3$ when $m = 3$ and an isomorphism $\mathbb{S}_4 \cong \mathbb{Q}^6 \subset \mathbb{P}^7$ when $m = 4$. Applying Lemma 5.11 with $m = 5$, we have a natural inclusion $\mathbb{S}_4 \cong \mathbb{S}^v \subset \mathbb{S}_5$ for any isotropic vector $v \in V$. From the isomorphism $\mathbb{S}_4 \simeq \mathbb{Q}^6$, it is an entry locus of $\mathbb{S}_5$. As these $\mathbb{S}^v$ cover $\mathbb{S}_5$, every entry locus of $\mathbb{S}_5$ is of the form $\mathbb{S}^v$ for some isotropic vector v. Take any two distinct entry loci $\mathbb{S}^v$ and $\mathbb{S}^{v'}$. If their intersection is non-empty, then $Q(v, v') = 0$ and $\mathbb{S}^v \cap \mathbb{S}^{v'} \simeq \mathbb{S}_3 \simeq \mathbb{P}^3$. $\square$

Recall (e.g. Proposition 1.3 (ii) in [IR1]) that the entry loci of a linear section with codimension $\leq \delta$ of a QEL-manifold S with $\delta \geq 1$ come from the corresponding linear section of entry loci of S. Thus Proposition 5.12 implies the following.

Corollary 5.13. *The intersection of any two distinct entry loci of a nonsingular linear section $\mathbb{S}_5$ with codimension $t \leq 2$ is either empty or isomorphic to $\mathbb{P}^k$ with $k \geq 3 - t$.*

6. Proof of Theorem 2.15

In this section, we will prove Theorem 2.15 by showing that a pair $(S \subset \mathbb{P}U, Y \subset \mathbb{P}W)$ determined by a special system of quadrics must be one of the examples in Example 2.3, Example 2.4, Proposition 2.11 or Proposition 2.12. We will divide our argument into a number of subsections. In the first two subsections, we will classify a certain type of special quadratic manifolds. In the next three subsections, we carry out the classification of the pair (S, Y).

6.1. **Classification when $\delta \geq \frac{1}{2} \dim S$.** As an easy application of Proposition 4.7 and Proposition 4.8, we have the following.

Proposition 6.1. *A d-dimensional special quadratic manifold $S^d \subset \mathbb{P}U$ with secant defect $\delta \geq \frac{d}{2}$ is projectively equivalent to one of the following. For our later use, we specify one possibility of $Y = Y(\sigma)$ in each case in the list.*

(i) *the smooth hyperquadric $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ with Y a point;*
(ii) *the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ with $Y = \mathbb{P}^2$;*
(iii) *the Plücker embedding of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ with $Y = \mathbb{P}^4$;*
(iv) *the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$ with $Y = \mathbb{Q}^8$;*
(v) *a general hyperplane section of (iii) with $Y = \mathbb{P}^4$;*
(vi) *a general hyperplane section of (ii) with $Y = \mathbb{P}^2$;*

(vii) *a general codimension 2 linear section of (iii) with* $Y = \mathbb{P}^4$*;*
(viii) *the Segre fourfold* $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ *with* $Y = \mathrm{Gr}(2,4) \simeq \mathbb{Q}^4$*.*

Proof. If $\delta = d$, the fact $\mathrm{Sec}(S) = \mathbb{P}U$ and $\delta = 2d + 1 - \dim \mathrm{Sec}(S)$ implies that S is a hyperquadric. This gives case (i).

As S is a QEL-manifold by Proposition 4.2, if $d > \delta \geq d/2$, we can apply Proposition 4.7 and Proposition 4.8. In Proposition 4.7, (A5) is excluded by Proposition 2.14. Thus only (A1)-(A4) are possible, which gives (ii)-(v).

In Proposition 4.8, the four Severi varieties (B2), (B4), (B7) and (B8) do not satisfy $\mathrm{Sec}(S) = \mathbb{P}U$ of Proposition 3.3. Also (B6) is excluded by Proposition 2.14. Thus only (B1), (B5) and (B3) are possible, which gives (vi)-(viii).

The example of $Y(\sigma)$ in each case follows from the table in Example 2.3, Proposition 2.11 and Proposition 2.12. □

6.2. **Classification when $\delta \geq 1$ and S is not a prime Fano manifold.** Special quadratic manifolds with $\delta \geq 1$ that are not prime Fano manifolds can be classified as follows.

Proposition 6.2. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with $\delta \geq 1$. If S is not a prime Fano manifold, then $S \subset \mathbb{P}U$ is projectively equivalent to one of the following:*

(c1) *a smooth conic in* $\mathbb{P}^2$*;*
(c2) *a general hyperplane section of the Segre 3-fold* $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$*;*
(c3) *the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$*.*

The proof is a direct consequence of the following result of Ionescu and Russo.

Theorem 6.3 (Theorem 2.2 in [IR1])**.** *Let $S \subset \mathbb{P}U$ be a nondegenerate, linearly normal, conic-connected manifold of dimension d. Then either S is a prime Fano manifold or it is projectively equivalent to one of the following:*

(C1) *the second Veronese embedding of* $\mathbb{P}^d$*.*
(C2) *the Segre embedding of* $\mathbb{P}^a \times \mathbb{P}^{d-a}$ *for* $1 \leq a \leq d-1$*.*
(C3) *the VMRT of the symplectic Grassmannian* $\mathrm{Gr}_\omega(k, k+d+1)$ *for* $2 \leq k \leq d$*.*
(C4) *a hyperplane section of the Segre embedding* $\mathbb{P}^a \times \mathbb{P}^{d+1-a}$ *with* $2 \leq a, d+1-a$*.*

Proof of Proposition 6.2. By Proposition 4.2, we know that $S \subset \mathbb{P}U$ is linearly normal. By Theorem 2.1 of [R2], the condition $\delta \geq 1$ implies that S is conic-connected. Thus if S is not a prime Fano manifold, it must be one of (C1)-(C4) in Theorem 6.3.

Recall that we have $\mathrm{Sec}(S) = \mathbb{P}U$ by Proposition 3.3. It is well known (e.g. the table in p.466 of [FH]) that only (c1) in (C1) and (c2) in (C2) satisfy $\mathrm{Sec}(S) = \mathbb{P}U$ and no cases in (C4) satisfy $\mathrm{Sec}(S) = \mathbb{P}U$. In (C3), only a general hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^d \subset \mathbb{P}^{2d+1}$ satisfies $\mathrm{Sec}(S) = \mathbb{P}U$ by Lemma 4.19 [FH]. By Proposition 2.14, only (c3) can occur. □

6.3. **Classification when $Y = \mathbb{P}^{2(c-1)}$.** We can classify special quadratic manifolds in the case of (Y1) of Theorem 3.10:

Proposition 6.4. *Assume that $Y = \mathbb{P}^{2(c-1)}$, then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:*

(i) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$;
(ii) $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
(iii) $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$;
(iv) *a general hyperplane section of (ii);*
(v) *a general codimension ≤ 2 linear section of (iii);*
(vi) *a general codimension-3 linear section of (iii);*
(vii) *a general codimension-2 linear section of (ii).*

All of the above cases do occur with $Y = \mathbb{P}^{2(c-1)}$ from Example 2.3, Proposition 2.11 and Proposition 2.12.

Proof. As $Y = \mathbb{P}^{2(c-1)}$, we have $\chi(Y) = 2c - 1$. Proposition 3.9 yields $\chi(S) = 2\delta + 2$. By Corollary 3.11, we obtain $2\delta + 2 \geq d + 1$, i.e., $\delta \geq (d-1)/2$. If $\delta \geq d/2$, then Proposition 6.1 gives (i)-(v). It remains to handle the case $\delta = (d-1)/2$ and $\chi(S) = d+1$. The latter condition combined with Corollary 3.11 implies that all even Betti numbers of S must be 1. We will argue case-by-case, depending on the values of δ.

(1) If $\delta \geq 3$, then $d - \delta = \delta + 1$ is divisible by $2^{[\frac{\delta-1}{2}]}$ by Proposition 4.6. This implies that either $(\delta, d, n) = (3, 7, 13)$ or $(\delta, d, n) = (7, 15, 25)$.

(1a) When $(\delta, d, n) = (3, 7, 13)$, Proposition 4.6 shows that S is a prime Fano manifold with $K_S^{-1} = \mathcal{O}(\frac{d+\delta}{2}) = \mathcal{O}(5)$. By Proposition 4.10, the only possibility is (M2), a linear section of $\mathbb{S}_5$. But such a case can not occur by Proposition 2.14.

(1b) When $(\delta, d, n) = (7, 15, 25)$, we see that S is a prime Fano manifold and its VMRT $\mathcal{C}_x \subset \mathbb{P}^{14}$ at a general point $x \in S$ is a QEL-manifold of dimension $(d+\delta)/2 - 2 = 9$ and secant defect $\delta - 2 = 5$ from Proposition 4.6. It follows that the subvariety $\mathrm{Locus}(x) \subset S$ has dimension 10 and has nonempty intersection with a general entry locus $\Sigma \subset S$ of dimension 7, because all even Betti numbers of S are

1. Since S is defined by quadrics and satisfies Condition 5.2 in Section 5 by Lemma 5.3, we can apply Theorem 5.10 to conclude that two general entry loci Σ_1 and Σ_2 of S through a general point $x \in S$ intersect along $\mathbb{P}^2$. By Proposition 4.6, the lines on Σ_1 and Σ_2 through x give two entry loci on $\mathcal{C}_x \subset \mathbb{P}^{14}$. The intersection of these entry loci of $\mathcal{C}_x$ must be $\mathbb{P}^1$ corresponding to the lines through x in $\Sigma_1 \cap \Sigma_2 \cong \mathbb{P}^2$. On the other hand, applying Proposition 4.6 to $\mathcal{C}_x$, we see that $\mathcal{C}_x$ is a prime Fano manifold with $K_{\mathcal{C}_x}^{-1} = \mathcal{O}(7)$. Proposition 4.10 shows that $\mathcal{C}_x \subset \mathbb{P}^{14}$ is projectively equivalent to a codimension 1 linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$. Thus the intersection of any two distinct entry loci of $\mathcal{C}_x$ must be either empty or $\mathbb{P}^k$ with $k \geq 2$, from Corollary 5.13, a contradiction.

(2) If $\delta = 2$, then $d = 5$ and $n = 10$. As $d + \delta$ is odd, S is not a prime Fano manifold by Proposition 4.6. Applying Proposition 6.2, we see that $S \subset \mathbb{P}^9$ is the Segre 5-fold $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, a contradiction to $\chi(S) = d + 1$.

(3) If $\delta = 1$, then $d = 3$ and $n = 7$. Since none of the varieties in the list in Proposition 6.2 have $(d, n) = (3, 7)$, we see that S is a prime Fano manifold. Thus $K_S^{-1} = \mathcal{O}(2)$ by Proposition 4.6 (1). From the classification of prime Fano threefolds with $K_S^{-1} = \mathcal{O}(2)$ (e.g. 12.1 in [IP]), S is a general codimension-3 linear section of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$, which gives case (vi).

(4) If $\delta = 0$, then $d = 1$ and $n = 4$. Thus S is a twisted cubic by Proposition 4.12, which is (vii).

□

6.4. **Classification when $Y = \mathbb{Q}^{2(c-1)}$.** We can classify special quadratic manifolds in the case of (Y2) of Theorem 3.10:

Proposition 6.5. *For a special quadratic manifold $S \subset \mathbb{P}^{n-1}$ of codimension c, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following*

(i) *the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;*

(ii) *the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$.*

Both cases do occur with $Y = \mathbb{Q}^{2(c-1)}$ from Example 2.3.

Proof. If $\delta \geq \frac{d}{2}$, then Proposition 6.1 gives (i) and (ii).

To show that $\delta < \frac{d}{2}$ cannot occur, we may assume $d = 2\delta + 2$ and $n = 2d + 2 - \delta = 3\delta + 6$ by Proposition 3.12 and derive a contradiction. Under this assumption, putting $\chi(\mathbb{Q}^{2(c-1)}) = 2c$ in Proposition 3.9, we have $\chi(S) = d + 1$, which combined with Corollary 3.11 implies that

all even Betti numbers of S must be 1. We will argue case-by-case depending on the value of δ.

(1) Assume $\delta \geq 3$. Proposition 4.6 shows that the VMRT $\mathcal{C}_x \subset \mathbb{P}^{d-1}$ at a general point $x \in S$ is a QEL-manifold of dimension $\frac{1}{2}(d+\delta-4) = \frac{1}{2}(3\delta-2)$ and secant defect $\delta - 2$. Since $d - \delta = \delta + 2$ is divisible by $2^{[\frac{\delta-1}{2}]}$ from Proposition 4.6, we have two possibilities: $(\delta, d, n) = (4, 10, 18)$ or $(\delta, d, n) = (6, 14, 24)$.

(1a) When $(\delta, d, n) = (4, 10, 18)$, the VMRT $\mathcal{C}_x \subset \mathbb{P}^9$ is a QEL manifold of dimension 5 and secant defect 2. By Proposition 4.6, this implies that $\mathcal{C}_x \subset \mathbb{P}^9$ is a not a prime Fano manifold. Thus Proposition 6.2 shows that $\mathcal{C}_x \subset \mathbb{P}^9$ is projectively equivalent to the Segre 5-fold $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$. By Main Theorem of Section 2 in [Mo], S is isomorphic to $\mathrm{Gr}(2, 7)$. But $S \subset \mathbb{P}^{17}$ is linearly normal by Proposition 4.2, which is not possible for $S = \mathrm{Gr}(2, 7)$.

(1b) When $(\delta, d, n) = (6, 14, 24)$, we see that S is a prime Fano manifold and its VMRT $\mathcal{C}_x \subset \mathbb{P}^{13}$ at a general point $x \in S$ is a QEL-manifold of dimension 8 and secant defect 4 from Proposition 4.6. It follows that the subvariety $\mathrm{Locus}(x) \subset S$ has dimension 9 and has nonempty intersection with a general entry locus $\Sigma \subset S$ of dimension 6, because all even Betti numbers of S are 1. Since S is defined by quadrics and satisfies Condition 5.2 in Section 5 by Lemma 5.3, we can apply Theorem 5.10 to conclude that two general entry loci Σ_1 and Σ_2 of S through a general point $x \in S$ intersect along $\mathbb{P}^1$. By Proposition 4.6, the lines on Σ_1 and Σ_2 through s give two entry loci on $\mathcal{C}_x \subset \mathbb{P}^{13}$. The intersection of these entry loci of $\mathcal{C}_x$ must be one point corresponding to the unique line through x in $\Sigma_1 \cap \Sigma_2$. On the other hand, applying Proposition 4.6 to $\mathcal{C}_x$, we see that $\mathcal{C}_x$ is a prime Fano manifold with $K^{-1}_{\mathcal{C}_x} = \mathcal{O}(6)$. Proposition 4.10 shows that $\mathcal{C}_x \subset \mathbb{P}^{13}$ is projectively equivalent to a codimension 2 linear section of $\mathbb{S}_{10} \subset \mathbb{P}^{15}$. Thus the intersection of any two distinct entry loci of $\mathcal{C}_x$ must be either empty or $\mathbb{P}^k$ with $k \geq 1$ from Corollary 5.13, a contradiction.

(2) If $\delta = 2$, then $d = 6$ and $n = 12$.

(2a) If S is not a prime Fano manifold, then Proposition 6.2 shows that S is the Segre 6-fold $\mathbb{P}^1 \times \mathbb{P}^5 \subset \mathbb{P}^{11}$. This contradicts $\chi(S) = d + 1$.

(2b) If S is a prime Fano manifold, then Proposition 4.10 is applicable. The only possibility is $s = 4$ of (M2), i.e., a

codimension-4 linear section of the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$. But this can not be a special quadratic manifold by Proposition 2.14.

(3) If $\delta = 1$, then $d = 4$ and $n = 9$. As $d + \delta$ is odd, Proposition 4.6 shows that S is not a prime Fano manifold. This is not possible by Proposition 6.2.

(4) If $\delta = 0$, then $d = 2$ and Proposition 4.12 is applicable. But the surfaces in Proposition 4.12 do not have Picard number 1, a contradiction.

□

6.5. **Classification when $Y = \mathrm{Gr}(2, c+1)$.** To handle the case (Y3) of Theorem 3.10, we use the following result.

Proposition 6.6 (Theorem 2.17 ($b = 2$) [AS])**.** *Let V be a vector space with $\dim V = 2k + 1 \geq 7$. Let $\mu : \mathrm{Sym}^2 V \to W$ be a system of quadrics such that the base locus subscheme $B(\mu) \subset \mathbb{P}V$ is irreducible, nonsingular and nondegenerate. Let $\nu : \mathbb{P}V \dashrightarrow \mathbb{P}W$ be the associated rational map. Suppose that ν is generically injective and its proper image $\mathrm{Im}(\nu)$ is biregular to $\mathrm{Gr}(2, k+2)$. Then $B(\mu) \subset \mathbb{P}V$ is one of the rational normal scrolls. In particular, $B(\mu)$ has Picard number 2 and is covered by lines.*

We can classify special quadratic manifolds in the case of (Y3) of Theorem 3.10:

Proposition 6.7. *Let $S \subset \mathbb{P}^{n-1}$ be a special quadratic manifold of codimension $c \geq 4$ with $Y \subset \mathbb{P}W$ being an isomorphic projection of the Plücker variety $\mathrm{Gr}(2, c+1) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.*

Proof. Let $\sigma : \mathrm{Sym}^2 U \to W, \dim U = n$, be the special system of quadrics such that our S is the base locus $B(\sigma) \subset \mathbb{P}U$. Recall that a general fiber of $\psi : \mathrm{Bl}_S(\mathbb{P}U) \to Y(\sigma) \cong \mathrm{Gr}(2, c+1)$ is sent to a linear space of dimension $\delta + 1$ in $\mathbb{P}U$ by Proposition 3.8. Take a general subspace $V \subset U$ with

$$\dim U - \dim V = \delta + 1, \ \dim V = 2c - 1 \geq 7$$

and denote by $S' \subset \mathbb{P}V$ the linear section $S \cap \mathbb{P}V$. Then the restriction

$$\psi|_{\mathrm{Bl}_{S'}(\mathbb{P}V)} : \mathrm{Bl}_{S'}(\mathbb{P}V) \to Y(\sigma)$$

is birational. The birational map $\mathbb{P}V \dashrightarrow Y(\sigma)$ is given by a system of quadrics $\mu : \mathrm{Sym}^2 V \to W$ induced by the restriction of σ. Since the

base locus $B(\mu)$ is S', which is irreducible, nonsingular and nondegenerate in $\mathbb{P}V$, we can apply Proposition 6.6 to see that S' is a rational normal scroll. We will argue case-by-case as follows.

(1) When $\delta = 0$. Putting $n = 2d + 2$ and $\chi(\mathrm{Gr}(2, c + 1)) = \frac{c(c+1)}{2}$ in Proposition 3.9, we have $\chi(S) = d + 1$. This implies that S has Picard number 1.
  - (1a) If $c \geq 5$, then $\dim S' = c - 2 \geq 3$. Thus S' and S have the same Picard number by Lefschetz. Since S' has Picard number 2, this gives a contradiction.
  - (1b) If $c = 4$, then $d = 3$. We conclude that $S \subset \mathbb{P}^7$ is a Fano threefold of Picard number 1 with $\chi(S) = 4$. Furthermore, a general hyperplane section S' of S is a rational normal scroll. In particular, S is covered by lines, which implies that $K_S^{-1} = \mathcal{O}(k), k \geq 2$. From the classification of Fano threefolds (e.g. 12.1 of [IP]), there are three possibilities for such Fano threefolds: $\mathbb{P}^3, \mathbb{Q}^3$ and a codimension-3 linear section of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$. But none of these can have $\delta = 0$.

(2) When $\delta \geq 1$.
  - (2a) Assume that S is a prime Fano manifold and $c \geq 5$. Then $\dim S' = c - 2 \geq 3$, which implies that S' and S have the same Picard number by Lefschetz. Since S' has Picard number 2, this gives a contradiction.
  - (2b) Assume that S is a prime Fano manifold and $c = 4$. Then $n = d + 5$ and $d - \delta = 3$ is odd. This is a contradiction to Proposition 4.6.
  - (2c) Assume that S is not a prime Fano manifold. Proposition 6.2 shows that $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.

□

7. Quadratically symmetric varieties and prolongations

In this section, we give an intrinsic characterization of $Z(\sigma)$ and relate it to the problem studied in [FH]. Throughout this section, $X \subset \mathbb{P}V$ denotes an n-dimensional irreducible nondegenerate projective subvariety.

Definition 7.1. Let $X \subset \mathbb{P}V$ be an n-dimensional irreducible nondegenerate projective subvariety. We say that X is a *quadratically symmetric variety* if there exists a Zariski open subset $X^o \subset \mathrm{Sm}(X)$ in the nonsingular locus of X such that for each $P \in X^o$, there exits a linear representation of the $\mathbb{C}^*$-group denoted by $E_P : \mathbb{C}^* \times V \to V$ on V satisfying the following conditions.

(i) Denoting by $\widehat{P} \subset V$ the tautological line over P and by $T_P(\widehat{X}) \subset V$ the affine tangent space, the $\mathbb{C}^*$-action E_P has weight 0 on $\widehat{P}$, weight 1 on $T_P(\widehat{X})/\widehat{P}$ and weight 2 on $V/T_P(\widehat{X})$. In other words, there exist subspaces $T'_P \subset T_P(\widehat{X})$ and $N'_P \subset V$ with

$$V = \widehat{P} \oplus T'_P \oplus N'_P \text{ and } T_P(\widehat{X}) = \widehat{P} \oplus T'_P$$

such that $\lambda \in \mathbb{C}^*$ acts by

$$E^{\lambda}_P(t \in \widehat{P}, u \in T'_P, w \in N'_P) = (t, \lambda u, \lambda^2 w).$$

(2) The induced $\mathbb{C}^*$-action on $\mathbb{P}V$ preserves X.

Notation 7.2. For a projective manifold $M \subset \mathbb{P}^N$, the second fundamental form at a point $x \in M$ is the non-empty linear system of quadrics on the tangent space T_xM, denoted by $|\mathrm{II}_{x,M}| \subset \mathbb{P}(\mathrm{Sym}^2 T_xM)^*$, which comes from the derivative of the Gauss map (see Section 3.2 of [IL] or p.602 of [R2] for details).

Proposition 7.3. *Let $X \subset \mathbb{P}V, \dim V = N + 1$, be a quadratically symmetric variety of dimension n. For a fixed $P \in X^o$ and the representation E_P of Definition 7.1, let $(x_0, x_1, \ldots, x_N)$ be linear coordinates on V such that the dual action of $\lambda \in \mathbb{C}^*$, denoted by the same symbol E^{λ}_P for simplicity, is given by $E^{\lambda}_P(x_0) = x_0$ and*

$$E^{\lambda}_P(x_i) = \begin{cases} \lambda x_i & \text{if } 1 \le i \le n \\ \lambda^2 x_i & \text{if } n+1 \le i \le N. \end{cases}$$

Let $z_i = \frac{x_i}{x_0}, 1 \le i \le N$, be the inhomogeneous coordinates on $\mathbb{P}V$ centered at P. View the germ of $X \subset \mathbb{P}V$ at P as the graph of the germ of a holomorphic map

$$z_k = F^k(z_1, \ldots, z_n), n+1 \le k \le N.$$

Then $F^k(z_1, \ldots, z_n)$ are homogeneous quadratic polynomials and the system of quadrics on T_PX defined by $\{F^k, n+1 \le k \le N\}$ under the identification of $T^*_PX = \mathbb{C}z_1 + \cdots + \mathbb{C}z_n$ is isomorphic to the second fundamental form $|\mathrm{II}_{P,X}|$ of X at P.

Proof. It is well-known (e.g. p. 108 of [IL]) that the quadratic terms in the Taylor expansion of F^k's give the second fundamental form $|\mathrm{II}_{P,X}|$. So it suffices to show that F^k's are quadratic. The $\mathbb{C}^*$-action E_P induces a $\mathbb{C}^*$-action on the affine space $\mathbb{P}V \setminus (x_0 = 0) = \{(z_1, \ldots, z_N)\}$ by

$$E^{\lambda}_P(z_i) = \begin{cases} \lambda x_i & \text{if } 1 \le i \le n \\ \lambda^2 z_i & \text{if } n+1 \le i \le N. \end{cases}$$

Since this action preserves the germ of X at P, the equations

$$z_k = F^k(z_1, \ldots, z_n), n+1 \le k \le N$$

must remain unchanged under the $\mathbb{C}^*$-action. Thus the Taylor expansion of F^k can have only quadratic terms. $\square$

Proposition 7.4. *Let X_1 and X_2 be two quadratically symmetric varieties of the same dimension in $\mathbb{P}V$. Assume that there are points $P_1 \in X_1^o$ and $P_2 \in X_2^o$ such that the second fundamental forms II_{P_1,X_1} and II_{P_2,X_2} are isomorphic as systems of quadrics. Then X_1 and X_2 are projectively equivalent.*

Proof. We can choose the inhomogeneous coordinates in Proposition 7.3 such that the germ of $X_i \subset \mathbb{P}V$ at P_i for $i = 1, 2$ is defined by equations $z_k = F_i^k(z_1, \ldots, z_n)$ for some quadratic polynomials $F_i^k, n + 1 \leq m, i = 1, 2$. Since II_{P_1,X_1} and II_{P_2,X_2} are isomorphic, we can assume that $F_1^k = F_2^k$ by linear coordinate changes. It follows that X_1 and X_2 are projectively equivalent. $\square$

The following is a generalization of $Z(\sigma)$ in Definition 2.1.

Notation 7.5. Let U, W be two vector spaces and let $\sigma : \mathrm{Sym}^2 U \to W$ be a surjective homomorphism. Fix a 1-dimensional vector space T with a fixed identification $T = \mathbb{C}$. Define a rational map $\phi^o : \mathbb{P}(T \oplus U) \dashrightarrow \mathbb{P}(T \oplus U \oplus W)$ by

$$[t : u] \mapsto [t^2 : tu : \sigma(u, u)] \text{ for } t \in T, u \in U.$$

The proper image of $\mathbb{P}(T \oplus U)$ under ϕ^o will be denoted by $Z = Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$. Note that ϕ sends $U \cong \mathbb{P}(T \oplus U) \setminus \mathbb{P}U$ isomorphically to $Z \setminus \mathbb{P}(U \oplus W)$. Thus Z is a rational variety and $\phi^o : \mathbb{P}(T \oplus U) \dashrightarrow Z$ is a birational map.

Proposition 7.6. *In Notation 7.5, the variety $Z(\sigma)$ is a quadratically symmetric variety.*

Proof. Consider the $\mathbb{C}^*$-action on $\mathbb{P}(T \oplus U \oplus W)$ given by $\lambda \cdot [t : u : w] = [t : \lambda u : \lambda^2 w]$ for all $\lambda \in \mathbb{C}^*$. Then it preserves $Z(\sigma)$. This gives E_P in Definition 7.1 when $P = \mathbb{P}T \in Z(\sigma)$.

As in the proof of Proposition 2.6, we can associate to each vector $v \in U$ the linear automorphism g_v of $\mathbb{P}(T \oplus U \oplus W)$ defined by

$$g_v : [t : u : w] \mapsto [t : u + tv : w + 2\sigma(u, v) + t\sigma(v, v)].$$

Then g_v preserves $Z(\sigma)$ as we saw in the proof of Proposition 2.6. In particular, the linear automorphism group of $Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$ acts transitively on $Z(\sigma) \setminus \mathbb{P}(U \oplus W)$. Thus E_P exists for any $P \in Z(\sigma) \setminus \mathbb{P}(U \oplus W)$. $\square$

Proposition 7.7. *Any quadratically symmetric variety $X \subset \mathbb{P}V$ is projectively equivalent to $Z(\sigma)$, where $\sigma : \mathrm{Sym}^2 T_P X \to (T_P\mathbb{P}V)/T_P X$ is the dual of the second fundamental form of X at a point $P \in X^o$.*

Proof. By Section 3 of [L], the second fundamental form of $Z(\sigma)$ at a point in $Z(\sigma) \setminus \mathbb{P}(U \oplus W)$ is isomorphic to σ. Hence X is projectively equivalent to $Z(\sigma)$ by Proposition 7.4 and Proposition 7.6. □

Now we can given an intrinsic characterize the projective manifold $Z(\sigma)$ associated to a special system of quadrics σ as follows.

Theorem 7.8. *Let $X \subset \mathbb{P}V$ be a prime Fano manifold. Then $X \subset \mathbb{P}V$ is a quadratically symmetric variety if and only if $X = Z(\sigma)$ for a special system of quadrics σ. In particular, Theorem 2.15 gives a classification of quadratically symmetric prime Fano manifolds.*

Proof. From Theorem 2.15 and Proposition 7.6, the manifold $Z(\sigma)$ associated to a special system of quadrics σ is a quadratically symmetric prime Fano manifold.

Conversely, a quadratically symmetric prime Fano manifold is of the form $Z(\sigma)$ by Proposition 7.7 where σ is its second fundamental form. By Proposition 2.15 of [AS], the locus $\sigma(u, u) = 0$ is exactly the VMRT at a general point of X. Since $Z(\sigma)$ is a prime Fano manifold, its VMRT at a general point is smooth and irreducible by Proposition 6.6 of [FH]. This implies that $X \subset \mathbb{P}V$ is $Z(\sigma)$ associated to a special system of quadratics σ. □

Theorem 7.8 enables us to relate special quadratic manifolds to the problem studied in [FH]. Let us recall the basic definitions.

Definition 7.9. Let V be a complex vector space and $\mathfrak{g} \subset \mathrm{End}(V)$ a Lie subalgebra. The *first prolongation* (denoted by $\mathfrak{g}^{(1)}$) of $\mathfrak{g}$ is the space of bilinear homomorphisms $A : \mathrm{Sym}^2 V \to V$ such that for any fixed $w \in V$, the endomorphism $A_w : V \to V$ defined by

$$v \in V \mapsto A_{w,v} := A(w, v) \in V$$

is in $\mathfrak{g}$. In other words, $\mathfrak{g}^{(1)} = \mathrm{Hom}(\mathrm{Sym}^2 V, V) \cap \mathrm{Hom}(V, \mathfrak{g})$.

Definition 7.10. Let $X \subset \mathbb{P}V$ be an irreducible subvariety. Denote by $\widehat{X} \subset V$ the affine cone of X and by $T_\alpha(\widehat{X}) \subset V$ the tangent space at a smooth point $\alpha \in \widehat{X}$. The Lie algebra of infinitesimal linear automorphisms of $\widehat{X}$ is

$$\mathfrak{aut}(\widehat{X}) := \{g \in \mathrm{End}(V) | g(\alpha) \in T_\alpha(\widehat{X}) \text{ for any smooth point } \alpha \in \hat{X}\}.$$

We will call $\mathfrak{aut}(\widehat{X})^{(1)}$ the *prolongation* of $X \subset \mathbb{P}V$. We say that X *has nonzero prolongation* if $\mathfrak{aut}(\widehat{X})^{(1)} \neq 0$.

Proposition 7.11. *Let $X \subset \mathbb{P}V$ be a quadratically symmetric variety. Then $\mathfrak{aut}(\widehat{X})^{(1)} \neq 0$.*

Proof. By Proposition 7.7, we may assume that $X = Z(\sigma) \subset \mathbb{P}V$, with $V = T \oplus U \oplus W$ associated to a surjective homomorphism $\sigma : \mathrm{Sym}^2 U \to W$. Define $A : \mathrm{Sym}^2 V \to V$ by

$$A\left((t_1, u_1, w_1), (t_2, u_2, w_2)\right) = \left(t_1 t_2, \frac{t_1 u_2 + t_2 u_1}{2}, \sigma(u_1, u_2)\right).$$

We claim that $A \in \mathfrak{aut}(\widehat{X})^{(1)}$ which proves the proposition.

To check the claim, we have to show that $A(v, \alpha) \in T_\alpha \widehat{X}$ for any $v \in V$ and a general smooth point α of $\widehat{X}$. It suffices to consider α in the open subset $\{(t, u, w) \mid t \neq 0, tw = \sigma(u, u)\}$ of $\widehat{X}$. Fix

$$\alpha = (t, u, w = \frac{1}{t}\sigma(u, u)) \in V, t \neq 0.$$

Then $T_\alpha \widehat{X}$ is the subspace of V consisting of (t', u', w') satisfying

$$(t + \epsilon t')(w + \epsilon w') = \sigma(u + \epsilon u', u + \epsilon u') \ \text{ modulo } \epsilon^2.$$

Thus

$$\begin{aligned} T_\alpha \widehat{X} &= \{(t', u', w') \in V \mid tw' + t'w - 2\sigma(u, u') = 0\} \\ &= \{(t', u', w') \mid tw' + \sigma(u, \frac{t'}{t}u - 2u') = 0\}. \end{aligned}$$

For any $v = (t_0, u_0, w_0)$, we have

$$A(v, \alpha) = A((t_0, u_0, w_0), (t, u, w)) = (t_0 t, \frac{t_0 u + t u_0}{2}, \sigma(u_0, u)).$$

Writing the right hand side as (t', u', w'), we have

$$tw' + \sigma(u, \frac{t'}{t}u - 2u') = t\sigma(u_0, u) + \sigma(u, \frac{t_0 t}{t}u - 2\frac{t_0 u + t u_0}{2}) = 0.$$

This proves the claim. $\square$

The following is a partial converse of Proposition 7.11.

Proposition 7.12. *Let $X \subset \mathbb{P}V$ be a nonsingular nondegenerate linearly normal projective variety. If X has nonzero prolongation, then X is a quadratically symmetric variety.*

Proof. This is contained in the proof of Theorem 1.1.3 [HM05]. In fact, it is shown in p. 606 of [HM05] that for a nonzero element $A : \mathrm{Sym}^2 V \to V$ of the prolongation of X and a general point $\alpha \in \widehat{X}$, the endomorphism A_α of V satisfies $A_\alpha^3 = 0$ and the semisimple part of the endomorphism A_α generates a $\mathbb{C}^*$-action with weight 0 on α and weight 1 on $T_\alpha X$. The orbits of this $\mathbb{C}^*$-action on X have degree ≤ 2 from $A_\alpha^3 = 0$. Since X is nondegenerate in $\mathbb{P}V$, this implies that the

weight on the complement of $T_\alpha \widehat{X}$ is 2. Thus X must be quadratically symmetric. □

Using the above results, we can derive from Theorem 2.15 the following classification result.

Theorem 7.13. *Let $X \subsetneq \mathbb{P}V$ be an irreducible nonsingular nondegenerate variety such that $\mathfrak{aut}(\widehat{X})^{(1)} \neq 0$. Then $X \subset \mathbb{P}V$ is projectively equivalent to one in the following list.*

(1) *The VMRT of an irreducible Hermitian symmetric space of compact type of rank ≥ 2 from Example 4.4.*
(2) *The VMRT of a symplectic Grassmannian from Example 4.5.*
(3) *A nonsingular linear section of* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ *of codimension* ≤ 2.
(4) *A $\mathbb{P}^4$-general linear section of $\mathbb{S}_5 \subset \mathbb{P}^{10}$ of codimension ≤ 3.*
(5) *Biregular projections of (1) and (2) with nonzero prolongation, which are completely described in Section 4 of* [FH].

Proof. The assumption $\mathfrak{aut}(\widehat{X})^{(1)} \neq 0$ implies that X is conic-connected by Theorem 1.1.3 [HM05].

If X is not a prime Fano manifold and X is linearly normal, then $X \subset \mathbb{P}V$ can be classified by Theorem 6.3. This is done in Proposition 6.4 of [FH]: they are exactly (1) and (2), excepting the prime Fano manifolds belonging to (1). Note that prime Fano manifolds belonging to (1) are exactly those appearing as $Z(\sigma)$ in Example 2.3.

If X is a prime Fano manifold and $X \subset \mathbb{P}V$ is linearly normal, then by Proposition 7.7 and Proposition 7.12, we see that $X \subset \mathbb{P}V$ is projectively equivalent to $Z(\sigma)$, where σ is the second fundamental form of X at a general point. Thus we have (3), (4) and prime Fano manifolds in (1), from Theorem 2.15 and the data on (1) and (2) in Section 3 of [FH].

Finally, if X is not linearly normal, it must be a biregular projection of the linearly normal one by Corollary 4.8 [FH]. (3) and (4) do not have biregular projections. So we have (5). □

Remark 7.14. Main Theorem in [FH] asserted the classification result in Theorem 7.13, but there was a flaw: the list there missed the case of (3) with codimension 2 and the cases of (4) with codimension 2 and 3. This omission is caused by Proposition 2.9 of [FH], where it was claimed that $\dim \mathfrak{aut}(\hat{X})^{(1)} \neq 1$ for any nonsingular nondegenerate linearly normal variety $X \subset \mathbb{P}V$. This proposition is incorrect: the codimension-2 case of (3) and the codimension-3 case of (4) satisfy $\dim \mathfrak{aut}(\hat{X})^{(1)} = 1$. The error in the proof of Proposition 2.9 occurred

at the very last line: the statement that the set of α and α' (in the notation therein) spans the vector space V is wrong.

References

[AS] A. Alzati and J. C. Sierra: Quadro-quadric special birational transformations of projective spaces, Int. Math. Res. Not. (2013), doi:10.1093/imrn/rnt173

[ES] L. Ein and N. Shepherd-Barron: Some special Cremona transformations, Amer. J. Math., **111** (1989), 783-800

[FH] B. Fu and J.-M. Hwang: Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity. Invent. math. **189** (2012) 457-513

[GH] P. Griffiths and J. Harris: Principles of Algebraic Geometry, Wiley, New York, 1978

[Ha] R. Hartshorne: Algebraic Geometry, Springer, New York, 1977

[HM05] J.-M. Hwang and N. Mok: Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation. Invent. Math. **160** (2005) 591-645

[IR1] P. Ionescu and F. Russo: Conic-connected manifolds, J. Reine Angew. Math. **644** (2010), 145-157

[IR2] P. Ionescu and F. Russo: Varieties with quadratic entry locus, II, Compos. Math. **144** (2008), 949-962.

[IP] V. A. Iskovskikh and Yu. G. Prokhorov: Fano varieties. Algebraic geometry, V, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.

[IL] T. A. Ivey and J. M. Landsberg: Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems, Graduate Studies in Mathematics, vol. 61 (2003)

[L] J. M. Landsberg: On second fundamental forms of projective varieties, Invent. Math. **117** (1994), 303-315

[Mo] N. Mok: Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents. Third International Congress of Chinese Mathematicians. Part 1, 2, 41–61, AMS/IP Stud. Adv. Math., 42, pt.1, 2, Amer. Math. Soc., Providence, RI, 2008

[M] S. Mukai: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, Proc. Natl. Acad. Sci. USA **86** (1989), 3000-3002

[P] B. Pasquier: On some smooth projective two-orbit varieties with Picard number 1. Math. Ann. **344** (2009), no. 4, 963-987.

[R1] F. Russo: On a theorem of Severi, Math. Ann. **316** (2000), 1-17

[R2] F. Russo: Varieties with quadratic entry locus, I, Math. Ann.**344** (2009), 597-617

[S] E. Sato: Projective manifolds swept out by large-dimensional linear spaces. Tôhoku Math. J. **49** (1997), no. 3, 299–321

[Z] F. L. Zak: *Tangents and secants of algebraic varieties*. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993

Baohua Fu
Institute of Mathematics, AMSS, Chinese Academy of Sciences,
55 ZhongGuanCun East Road, Beijing, 100190, China
bhfu@math.ac.cn

Jun-Muk Hwang
Korea Institute for Advanced Study, Hoegiro 85,
Seoul, 130-722, Korea
jmhwang@kias.re.kr